Cascade of phases in turbulent flows

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Abstract. This article is devoted to incompressible Euler equations (or to Navier-Stokes equations in the vanishing viscosity limit). It describes the propagation of *quasi-singularities*. The underlying phenomena are consistent with the notion of a *cascade of energy*.

Résumé. Cet article étudie les équations d'Euler incompressible (ou de Navier-Stokes en présence de viscosité évanescente). On y décrit la propagation de *quasi-singularités*. Les phénomènes sous-jacents confirment l'idée selon laquelle il se produit une *cascade d'energie*.

1 Introduction.

Consider incompressible fluid equations

$$(\mathcal{E}) \qquad \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} + \nabla \mathbf{p} = 0\,, \qquad \mathrm{div}\ \mathbf{u} = 0\,, \qquad (t,x) \in [0,T] \times \mathbb{R}^d\,,$$

where $\mathbf{u} = {}^t(\mathbf{u}^1, \cdots, \mathbf{u}^d)$ is the fluid velocity and $\mathbf{p}$ is the pressure. The structure of *weak* solutions of $(\mathcal{E})$ in $d-$space dimensions with $d \geq 2$ is a problem of wide current interest [4]-[23]. The questions are how to describe the phenomena with adequate models and how to visualize the results in spite of their complexity. We will achieve a small step in these two directions.

According to the physical intuition, the appearance of singularities is linked with the *increase of the vorticity.* Along this line, we have to mark the contributions [2] and [9]. Interesting objects are solutions which do not blow up in finite time but whose associated vorticities increase arbitrarily fast. These are *quasi-singularities.* Their study is of practical importance.

Typical examples of quasi-singularities are oscillations. This is a well-known fact going back to [3]-[24]. The works [3] and [24] rely on phenomenological considerations and engineering experiments. Further developments are related to homogenization [12]-[13], compensated compactness [11]-[16] and non linear geometric optics [6]-[7]-[8].

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In [11], DiPerna and Majda show the persistence of oscillations in three dimensional Euler equations. To this end, they select a parameter $\varepsilon \in]0,1]$ and look at

$$\mathbf{u}_s^\varepsilon(t,x) \,:=\, {}^t\big(\mathbf{g}(x_2,\varepsilon^{-1}\,x_2),0,\mathbf{h}\big(x_1-\mathbf{g}(x_2,\varepsilon^{-1}\,x_2)\,t,x_2,\varepsilon^{-1}\,x_2\big)\big) \tag{1.1}$$

where $\mathbf{g}(x_2,\theta)$ and $\mathbf{h}(x_1,x_2,\theta)$ are smooth bounded functions with period 1 in θ. They remark that the functions $\mathbf{u}_s^\varepsilon$ are exact smooth solutions of $(\mathcal{E})$ and they let ε goes to zero. Yet, this construction is of a very special form. First, it comes from shear layers (steady 2-D solutions) as

$$\tilde{\mathbf{u}}_s^\varepsilon(t,x) \,=\, \tilde{\mathbf{u}}_s^\varepsilon(0,x) \,=\, {}^t\big(\mathbf{g}(x_2,\varepsilon^{-1}\,x_2),0\big) \,\in\, \mathbb{R}^2\,.$$

Secondly, it involves a phase $\varphi_0(t,x) \equiv x_2$ which does not depend on ε. Of course, this is a common fact [10]-[18]-[19]-[26] when dealing with large amplitude high frequency waves. Nevertheless, this is far from giving a complete idea of what can happen.

Our aim in this paper is to develop a theory which allows to remove the two restrictions mentioned above. Fix $\flat = (l,N) \in \mathbb{N}^2$ where the integers l and N are such that $0 < l < N$. Introduce the *geometrical* phase

$$\varphi_g^\varepsilon(t,x) \,:=\, \varphi_0(t,x) \,+\, \textstyle\sum_{k=1}^{l-1} \varepsilon^{\frac{k}{l}}\; \varphi_k(t,x)\,.$$

In the section 2, we state the Theorem 2.1 which provides with *approximate* solutions $\mathbf{u}_\flat^\varepsilon$ defined on the interval $[0,T]$ with $T>0$ and having the form

$$\begin{aligned} \mathbf{u}_\flat^\varepsilon(t,x) \,&=\, {}^t(\mathbf{u}_\flat^{\varepsilon 1},\cdots,\mathbf{u}_\flat^{\varepsilon d})(t,x) \\ &=\, \mathbf{u}_0(t,x) \,+\, \textstyle\sum_{k=1}^N \varepsilon^{\frac{k}{l}}\; U_k\big(t,x,\varepsilon^{-1}\,\varphi_g^\varepsilon(t,x)\big) \end{aligned} \tag{1.2}$$

where the smooth profiles

$$U_k(t,x,\theta) \,=\, {}^t(U_k^1,\cdots,U_k^d)(t,x,\theta) \in \mathbb{R}^d\,, \qquad 1 \le k \le N\,,$$

are periodic functions of $\theta \in \mathbb{R}/\mathbb{Z}$. We assume that

$$\exists\,(t,x,\theta) \in [0,T]\times\mathbb{R}^d\times\mathbb{T}\,; \qquad \partial_\theta U_1(t,x,\theta) \neq 0.$$

We say that the family $\{\mathbf{u}_\flat^\varepsilon\}_\varepsilon$ is a *weak*, a *strong* or a *turbulent* oscillation according as we have respectively $l=1$, $l=2$ or $l \ge 3$.

The order of magnitude of the energy of the oscillations is $\varepsilon^{\frac{1}{l}}$. Compute the vorticities associated with the functions $\mathbf{u}_\flat^\varepsilon$. These are the skew-symmetric matrices $\Omega_\flat^\varepsilon = (\Omega_{\flat j}^{\varepsilon i})_{1\le i,j\le d}$ where

$$\begin{aligned} \Omega_{\flat j}^{\varepsilon i}(t,x) \,&:=\, (\partial_j\mathbf{u}_\flat^{\varepsilon i} - \partial_i\mathbf{u}_\flat^{\varepsilon j})(t,x) \\ &=\, \textstyle\sum_{k=1}^N \varepsilon^{\frac{k}{l}-1}\; (\partial_j\varphi_g^\varepsilon\;\partial_\theta U_k^i - \partial_i\varphi_g^\varepsilon\;\partial_\theta U_k^j)\big(t,x,\varepsilon^{-1}\,\varphi_g^\varepsilon(t,x)\big) \\ &\quad + (\partial_j\mathbf{u}_0^i - \partial_i\mathbf{u}_0^j)(t,x) \,+\, \textstyle\sum_{k=1}^N \varepsilon^{\frac{k}{l}}\; (\partial_j U_k^i - \partial_i U_k^j)\big(t,x,\varepsilon^{-1}\,\varphi_g^\varepsilon(t,x)\big)\,. \end{aligned}$$

The principal term in $\Omega_\flat^\varepsilon$ is of size $\varepsilon^{\frac{1}{l}-1}$. When $l \geq 2$, no uniform estimates are available on the family $\{\Omega_\flat^\varepsilon\}_{\varepsilon \in]0,1]}$. In particular, if $d=3$, there is no uniform control on the enstrophy

$$\int_0^T \int_{\mathbb{R}^3} \ |\omega_\flat^\varepsilon(t,x)|^2 \ dt\, dx \,, \qquad \omega_\flat^\varepsilon(t,x) \,:=\, (\nabla \wedge \mathbf{u}_\flat^\varepsilon)(t,x) \,\equiv\, \Omega_\flat^\varepsilon(t,x)\,.$$

We see here that strong and turbulent oscillations are examples of quasi-singularities. Observe that the expansion (1.2) involves a more complicated structure than in (1.1) though the corresponding regime is less singular.

The BKW analysis reveals that the phase shift φ_1 and the terms φ_k with $2 \leq k \leq l-1$ play different parts. The rôle of φ_1 is partly examined in the articles [6] and [7] which deal with the case $l=2$. When $l \geq 3$, the phenomenon to emphasize is the creation of the φ_k with $2 \leq k \leq l-1$. Indeed, suppose that

$$\varphi_2(0,\cdot) \equiv \ \cdots \ \equiv \varphi_{l-1}(0,\cdot) \equiv 0\,, \qquad l \geq 3\,.$$

Then, generically, we find

$$\exists\, t \in]0,T]\,; \qquad \varphi_2(t,\cdot) \not\equiv 0\,, \qquad \cdots\,, \qquad \varphi_{l-1}(t,\cdot) \not\equiv 0\,.$$

Now starting with *large* amplitude waves (this corresponds to the limit case $l=+\infty$) that is

$$\mathbf{u}_\infty^\varepsilon(0,x) \,=\, \sum_{k=0}^\infty \varepsilon^k \ U_k\big(0,x,\varepsilon^{-1}\,\varphi_0(0,x)\big)\,, \qquad \partial_\theta U_0 \not\equiv 0\,,$$

the description of $\mathbf{u}_\infty^\varepsilon(t,\cdot)$ on the interval $[0,T]$ with $T>0$ needs the introduction of an *infinite cascade* of phases φ_k. The scenario is the following. Oscillations of the velocity develop spontaneously in all the intermediate frequencies $\varepsilon^{\frac{k}{l}-1}$ and in all the directions $\nabla\varphi_k(t,x)$. This expresses *turbulent* features in the flow.

The family $\{\mathbf{u}_\flat^\varepsilon\}_{\varepsilon \in]0,1]}$ is $\varepsilon-$stratified [19] with respect to the phase φ_g^ε with in general $\varphi_g^\varepsilon \not\equiv \varphi_0$. The presence in φ_g^ε of the non trivial functions φ_k is necessary and sufficient to encompass the *geometrical* features of the propagation. It has various consequences which are detailed in the section 3. It brings informations about microstructures, compensated compactness and non linear geometric optics. It also confirms observations made in the statistical approach of turbulences [14]-[22].

The chapter 4 is devoted to the demonstration of Theorem 2.1. Because of *closure problems*, the use of the geometrical phase φ_g^ε does not suffice to perform the BKW analysis. Among other things, *adjusting phases* φ_k with $l \leq k \leq N$ must be incorporated in order to put the system of formal equations in a triangular form.

The expressions $\mathbf{u}^\varepsilon_\flat$ are not exact solutions of Euler equations, yielding small error terms $\mathbf{f}^\varepsilon_\flat$ as source terms. The matter is to know if there exists exact solutions which coincide with $\mathbf{u}^\varepsilon_\flat(0,\cdot)$ at time $t=0$, which are defined on $[0,T]$ with $T>0$, and which are close to the approximate divergence free solutions $\mathbf{u}^\varepsilon_\flat$. This is a problem of *stability.*

The construction of exact solutions requires a good understanding of the different mechanisms of amplifications which occur. In the subsection 5.1, we make a distinction between *obvious* and *hidden* instabilities.

The obvious instabilities can be detected by looking at the BKW analysis presented before. They imply the non linear instability of Euler equations (Proposition 5.1). They need to be absorbed a dependent change of variables which induces a defect of hyperbolicity. The hidden instabilities can be revealed by soliciting this lack of hyperbolicity. They require to be controled the addition of dissipation terms.

In the subsection 5.2, we look at incompressible fluids with anisotropic viscosity. This is the framework of [5] though we adopt a different point of view. We consider strong oscillations. We show (Theorem 5.1) that *exact* solutions corresponding to $\mathbf{u}^\varepsilon_{(2,N)}$ exist on some interval $[0,T]$ with $T>0$ independent on $\varepsilon\in\,]0,1]$.

Contents

1 Introduction. **1**

2 Euler equations in the variables (t,x). **5**
2.1 Notations. 5
2.2 Divergence free approximate solutions in (t,x). 7
2.3 End of the proof of Theorem 2.1. 9

3 The cascade of phases. **13**
3.1 Microstructures. 13
3.2 The geometrical phase. 14
3.3 Closure problems. 14
3.4 Compensated compactness. 15
3.5 Non linear geometric optics. 16
3.6 The statistical approach. 20
3.7 Phenomenological comparison. 21

4 Euler equations in the variables (t,x,θ). **23**
4.1 Preliminaries. 23
4.2 The BKW analysis. 25
4.3 Divergence free approximate solutions in (t,x,θ). 30

5 Stability of strong oscillations **34**
5.1 Various types of instabilities 34
5.2 Exact solutions 37

2 Euler equations in the variables (t,x).

The description of incompressible flows in turbulent regime is a delicate question. No systematic analysis is yet available. However, special approximate solutions with rapidly varying structure in space and time can be exhibited. Their construction is summarized in this chapter 2.

2.1 Notations.

• *Variables.* Let $T \in \mathbb{R}_*^+$. The time variable is $t \in [0,T]$. Let $d \in \mathbb{N} \setminus \{0,1\}$. The space variables are $(x,\theta) \in \mathbb{R}^d \times \mathbb{T}$ where $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. Mark the ball

$$B(0,R] := \big\{ x \in \mathbb{R}^d \, ; \; |x|^2 := \textstyle\sum_{i=1}^d x_i^2 \leq R \big\} , \qquad R \in \mathbb{R}^+ .$$

The state variables are the velocity field $u = {}^t(u^1, \cdots, u^d) \in \mathbb{R}^d$ and the pressure $p \in \mathbb{R}$. Given $(u,\tilde{u}) \in (\mathbb{R}^d)^2$, define

$$u \cdot \tilde{u} := \textstyle\sum_{i=1}^d u^i \, \tilde{u}^i , \qquad |u|^2 := u \cdot u , \qquad u \otimes \tilde{u} := (u^j \, \tilde{u}^i)_{1 \leq i,j \leq d} .$$

The symbol S_+^d is for the set of positive definite quadratic form on $\mathbb{R}^d$. An element $\mathfrak{q} \in S_+^d$ can be represented by some $d \times d$ matrix $(\mathfrak{q}_{ij})_{1 \leq i,j \leq d}$.

• *Functional spaces.* Distinguish the expressions $\mathbf{u}(t,x)$ which do not depend on the variable θ from the expressions $u(t,x,\theta)$ which depend on θ. The boldfaced type $\mathbf{u}$ is used in the first case whereas the letter u is employed in the second situation.

Note $C_b^\infty([0,T] \times \mathbb{R}^d)$ the space of functions in $[0,T] \times \mathbb{R}^d$ with bounded continuous derivatives of any order. Let $m \in \mathbb{N}$. The Sobolev space H^m is the set of functions

$$u(x,\theta) \; = \; \textstyle\sum_{k \in \mathbb{Z}} \mathbf{u}_k(x) \; e^{i\,k\,\theta}$$

such that

$$\| \, u \, \|_{H^m}^2 := \; \textstyle\sum_{k \in \mathbb{Z}} (1 + |k|^2)^m \; \int_{\mathbb{R}^d} (1 + |\xi|^2)^m \; |\hat{\mathbf{u}}_k(\xi)|^2 \; d\xi \; < \; \infty$$

where

$$\mathcal{F}(\mathbf{u})(\xi) \; = \; \hat{\mathbf{u}}(\xi) \; := \; (2\,\pi)^{-\frac{d}{2}} \; \int_{\mathbb{R}^d} e^{-i\,x \cdot \xi} \; \mathbf{u}(x) \; dx , \qquad \xi \in \mathbb{R}^d .$$

With these conventions, the condition $\mathbf{u} \in H^m$ means simply that

$$\| \, \mathbf{u} \, \|_{H^m}^2 := \; \int_{\mathbb{R}^d} (1 + |\xi|^2)^m \; |\hat{\mathbf{u}}(\xi)|^2 \; d\xi \; < \; \infty \, .$$

Define

$$H_T^m \; := \; \big\{ \, u \; ; \; \partial_t^j u \in L^2([0,T]; H^{m-j}) \, , \; \forall \, j \in \{0, \cdots, m\} \, \big\} ,$$

$$\mathcal{W}_T^m \; := \; \big\{ \, u \; ; \; u \in C^j([0,T]; H^{m-j}) \, , \; \forall \, j \in \{0, \cdots, m\} \, \big\} \, ,$$

with the corresponding norms

$$\parallel u \parallel^2_{H^m_T} := \textstyle\sum_{j=0}^m \int_0^T \; \parallel \partial_t^j u(t,\cdot) \parallel^2_{H^m} \; dt \, ,$$

$$\parallel u \parallel_{\mathcal{W}^m_T} := \sup_{t \in [0,T]} \; \textstyle\sum_{j=0}^m \parallel \partial_t^j u(t,\cdot) \parallel_{H^m} \; .$$

Consider also

$$\begin{array}{lll} H^m_\infty \; := \bigcap_{T \in \mathbb{R}^+} H^m_T \, , & H^\infty_T \; := \bigcap_{m \in \mathbb{N}} H^m_T \, , & H^\infty_\infty \; := \bigcap_{T \in \mathbb{R}^+} H^\infty_T \, , \\ \mathcal{W}^m_\infty \; := \bigcap_{T \in \mathbb{R}^+} \mathcal{W}^m_T \, , & \mathcal{W}^\infty_T \; := \bigcap_{m \in \mathbb{N}} \mathcal{W}^m_T \, , & \mathcal{W}^\infty_\infty \; := \bigcap_{T \in \mathbb{R}^+} \mathcal{W}^\infty_T \, . \end{array}$$

When $m = 0$, replace H^0 with L^2. Any function $u \in L^2$ can be decomposed according to

$$u(t,x,\theta) \; = \; \langle u \rangle (t,x) + u^*(t,x,\theta) \; = \; \bar{u}(t,x) + u^*(t,x,\theta)$$

where

$$\langle u \rangle (t,x) \; \equiv \; \bar{u}(t,x) \; := \; \textstyle\int_{\mathbb{T}} \, u(t,x,\theta) \; d\theta \, .$$

Let Γ be the symbol of any of the spaces H^m, H^m_T, $\mathcal{W}^m_T$, $\cdots$ defined before. In order to specify the functions with mean value zero, introduce

$$\Gamma^* \; := \; \{ \, u \in \Gamma \, ; \; \bar{u} \equiv 0 \, \} \, .$$

Mark also

$$\mathrm{supp}_x \, u^* \; := \; \text{closure of } \big\{ \, x \in \mathbb{R}^d \, ; \; \parallel u^*(x,\cdot) \parallel_{L^2(\mathbb{T})} \neq \; 0 \, \big\} \, .$$

- *Differential operators.* Note

$$\begin{array}{ll} \partial_t \equiv \partial_0 := \partial / \partial \, t \, , & \partial_\theta \equiv \partial_{d+1} := \partial / \partial \, \theta \, , \\ \partial_j := \partial / \partial \, x_j \, , & j \in \{1, \cdots, d\} \, , \\ \nabla := (\partial_1, \cdots, \partial_d) \, , & \Delta := \Delta_x + \partial^2_\theta = \partial^2_1 + \cdots + \partial^2_d + \partial^2_\theta \, . \end{array}$$

Let $u \in \mathcal{W}^\infty_T$. Define

$$u \cdot \nabla \; := \; u^1 \; \partial_1 + \cdots + u^d \; \partial_d \, ,$$

$$\mathrm{div} \; u \; := \; \partial_1 u^1 + \cdots + \partial_d u^d \, ,$$

$$\mathrm{div} \; (u \otimes \tilde{u}) \; := \; \textstyle\sum_{j=1}^d \; {}^t \big(\partial_j (u^j \, \tilde{u}^1) \, , \cdots , \partial_j (u^j \, \tilde{u}^d) \big) \in \mathbb{R}^d \, .$$

Employ the bracket $< \cdot, \cdot >_H$ for the scalar product in the Hilbert space H. Note $\mathcal{L}(E;F)$ the space of linear continuous applications $T : E \longrightarrow F$ where E and F are Banach spaces. The symbol $\mathcal{L}(E)$ is simply for $\mathcal{L}(E;E)$. Introduce the commutator

$$[A;B] \; := \; A \circ B - B \circ A \, , \qquad (A,B) \in \mathcal{L}(E)^2 \, .$$

Let $r \in \mathbb{Z}$. The operator T is in $\mathfrak{L}^r$ if

$$\parallel T \parallel_{\mathcal{L}(H_T^{m+r};H_T^m)} < \infty \,, \qquad \forall\, m \in \mathbb{N} \,.$$

Let $\varepsilon_0 > 0$. The family of operators $\{T^\varepsilon\}_\varepsilon \in \mathcal{L}(H_T^\infty)^{]0,\varepsilon_0]}$ is in $\mathfrak{UL}^r$ if

$$\sup_{\varepsilon \in]0,\varepsilon_0]} \quad \parallel T^\varepsilon \parallel_{\mathcal{L}(H_T^{m+r};H_T^m)} < \infty \,, \qquad \forall\, m \in \mathbb{N} \,.$$

Consider a family $\{f^\varepsilon\}_\varepsilon \in (\mathcal{W}_T^\infty)^{]0,\varepsilon_0]}$. We say that $\{f^\varepsilon\}_\varepsilon = \bigcirc(\varepsilon^r)$ if

$$\sup_{\varepsilon \in]0,\varepsilon_0]} \quad \varepsilon^{-r} \quad \parallel f^\varepsilon \parallel_{\mathcal{W}_T^m} < \infty \,, \qquad \forall\, m \in \mathbb{N} \,.$$

Given a family $\{\mathbf{f}^\varepsilon\}_\varepsilon \in (\mathcal{W}_T^\infty)^{]0,\varepsilon_0]}$, we say that $\{\mathbf{f}^\varepsilon\}_\varepsilon = \bigcirc(\varepsilon^r)$ if

$$\sup_{\varepsilon \in]0,\varepsilon_0]} \quad \varepsilon^{-r+m} \quad \parallel \mathbf{f}^\varepsilon \parallel_{\mathcal{W}_T^m} < \infty \,, \qquad \forall\, m \in \mathbb{N} \,.$$

Observe that the two preceding definitions have very different significations according as we use the letter f or the boldfaced type $\mathbf{f}$. In particular, the second inequalities correspond to $\varepsilon-$ stratified estimates. The families $\{f^\varepsilon\}_\varepsilon$ or $\{\mathbf{f}^\varepsilon\}_\varepsilon$ are $\bigcirc(\varepsilon^\infty)$ if they are $\bigcirc(\varepsilon^r)$ for all $r \in \mathbb{R}$.

2.2 Divergence free approximate solutions in (t, x).

• **A first result.** Select smooth functions

$$\mathbf{u}_{00} \in H^\infty \,, \qquad \varphi_{00} \in C^1(\mathbb{R}^d) \,, \qquad \nabla\varphi_{00} \in C_b^\infty(\mathbb{R}^d) \,.$$

Suppose that

$$\exists\, c > 0 \,; \qquad |\nabla\varphi_{00}(x)| \,\geq\, 2\, c \,, \qquad \forall\, x \in \mathbb{R}^d \,.$$

For $T > 0$ small enough, the equation $(\mathcal{E})$ associated with

$$\mathbf{u}_0(0, x) = \mathbf{u}_{00}(x) \,, \qquad \forall\, x \in \mathbb{R}^d$$

has a smooth solution $\mathbf{u}_0(t, x) \in \mathcal{W}_T^\infty$. Solve the eiconal equation

$$(ei) \qquad \partial_t\varphi_0 + (\mathbf{u}_0 \cdot \nabla)\, \varphi_0 = 0 \,, \qquad (t, x) \in [0, T] \times \mathbb{R}^d$$

with the initial data

$$\varphi_0(0, x) = \varphi_{00}(x) \,, \qquad \forall\, x \in \mathbb{R}^d \,.$$

If necessary, restrict the time T in order to have

$$(2.1) \qquad |\nabla\varphi_0(t, x)| \,\geq\, c \,, \qquad \forall\, (t, x) \in [0, T] \times \mathbb{R}^d \,.$$

Call $\Pi_0(t, x)$ the orthogonal projector from $\mathbb{R}^d$ onto the hyperplane

$$\nabla\varphi_0(t, x)^\perp \,:=\, \big\{\, u \in \mathbb{R}^d ;\; u \cdot \nabla\varphi_0(t, x) = 0 \,\big\} \,.$$

Theorem 2.1. *Select any* $\flat = (l, N) \in \mathbb{N}_*^2$ *such that* $0 < l\,(3 + \frac{d}{2}) \ll N$. *Consider the following initial data*

$$\begin{aligned} & U_{k0}^*(x,\theta) = \Pi_0(0,x)\, U_{k0}^*(x,\theta) \in H^\infty\,, \qquad 1 \leq k \leq N\,, \\ & \bar{U}_{k0}(x) \in H^\infty\,, \qquad 1 \leq k \leq N\,, \\ & \varphi_{k0}(x) \in H^\infty\,, \qquad 1 \leq k \leq l-1\,. \end{aligned}$$

First, there are finite sequences $\{U_k\}_{1 \leq k \leq N}$ *and* $\{P_k\}_{1 \leq k \leq N}$ *with*

$$U_k(t,x,\theta) \in \mathcal{W}_T^\infty\,, \qquad P_k(t,x,\theta) \in \mathcal{W}_T^\infty\,, \qquad 1 \leq k \leq N\,,$$

and a finite sequence $\{\varphi_k\}_{1 \leq k \leq l-1}$ *with*

$$\varphi_k(t,x) \in \mathcal{W}_T^\infty\,, \qquad 1 \leq k \leq l-1\,,$$

which are such that

$$\begin{aligned} & \Pi_0(0,x)\, U_k^*(0,x,\theta) = \Pi_0(0,x)\, U_{k0}^*(x,\theta)\,, \qquad 1 \leq k \leq N\,, \\ & \bar{U}_k(0,x) = \bar{U}_{k0}(x)\,, \qquad 1 \leq k \leq N\,, \\ & \varphi_k(0,x) = \varphi_{k0}(x)\,, \qquad 1 \leq k \leq l-1\,. \end{aligned}$$

Secondly, there is $\varepsilon_0 \in]0,1]$ *and correctors*

$$\mathbf{cu}_\flat^\varepsilon(t,x) \in \mathcal{W}_T^\infty\,, \qquad \mathbf{cp}_\flat^\varepsilon(t,x) \in \mathcal{W}_T^\infty\,, \qquad \varepsilon \in]0,\varepsilon_0]\,,$$

which give rise to families satisfying

$$\{\mathbf{cu}_\flat^\varepsilon\}_\varepsilon \,=\, \bigcirc(\varepsilon^{\frac{N}{l}-2})\,, \qquad \{\mathbf{cp}_\flat^\varepsilon\}_\varepsilon \,=\, \bigcirc(\varepsilon^{\frac{N}{l}})\,.$$

Then, all these expressions are adjusted so that the functions $\mathbf{u}_\flat^\varepsilon$ *and* $\mathbf{p}_\flat^\varepsilon$ *defined according to*

$$\begin{aligned} \mathbf{u}_\flat^\varepsilon(t,x) &:= \mathbf{u}_0(t,x) + \textstyle\sum_{k=1}^N \varepsilon^{\frac{k}{l}}\; U_k\big(t,x,\varepsilon^{-1}\,\varphi_g^\varepsilon(t,x)\big) + \mathbf{cu}_\flat^\varepsilon(t,x) \\ \mathbf{p}_\flat^\varepsilon(t,x) &:= \mathbf{p}_0(t,x) + \textstyle\sum_{k=1}^N \varepsilon^{\frac{k}{l}}\; P_k\big(t,x,\varepsilon^{-1}\,\varphi_g^\varepsilon(t,x)\big) + \mathbf{cp}_\flat^\varepsilon(t,x) \end{aligned} \tag{2.2}$$

where $\varphi_g^\varepsilon(t,x)$ *is the geometrical phase*

$$\varphi_g^\varepsilon(t,x) \,:=\, \varphi_0(t,x) \,+\, \textstyle\sum_{k=1}^{l-1} \varepsilon^{\frac{k}{l}}\; \varphi_k(t,x) \tag{2.3}$$

are approximate solutions of $(\mathcal{E})$ *on the interval* $[0,T]$. *More precisely*

$$\partial_t \mathbf{u}_\flat^\varepsilon + (\mathbf{u}_\flat^\varepsilon \cdot \nabla)\mathbf{u}_\flat^\varepsilon + \nabla \mathbf{p}_\flat^\varepsilon = \mathbf{f}_\flat^\varepsilon\,, \qquad \text{div}\ \mathbf{u}_\flat^\varepsilon = 0\,, \qquad \mathbf{f}_\flat^\varepsilon = \bigcirc(\varepsilon^{\frac{N}{l}-3-\frac{d}{2}})\,.$$

• Some comments.

Remark 2.2.1: In what follows, we suppose that U_1^* is non trivial. In other words, we start with some initial data satisfying

$$\exists\,(x,\theta) \in \mathbb{R}^d \times \mathbb{T}\,; \qquad U_1^*(0,x,\theta) = U_{10}^*(x,\theta) \neq 0\,. \tag{2.4}$$

$\triangle$

Remark 2.2.2: Fix any $l \in \mathbb{N}_*$. The Borel's summation process allows to take $N = +\infty$ in the Theorem 2.1. It yields BKW solutions $(\mathbf{u}^\varepsilon_\flat, \mathbf{p}^\varepsilon_\flat)$ which solve $(\mathcal{E})$ with infinite accuracy

$$\partial_t \mathbf{u}^\varepsilon_\flat + (\mathbf{u}^\varepsilon_\flat \cdot \nabla)\mathbf{u}^\varepsilon_\flat + \nabla \mathbf{p}^\varepsilon_\flat \; = \; \bigcirc(\varepsilon^\infty) \, , \qquad \text{div } \mathbf{u}^\varepsilon_\flat \; = \; 0 \, . \qquad \triangle$$

Remark 2.2.3: Suppose that the function $\mathbf{u}_0 \in \mathcal{W}^\infty_\infty$ is a global solution of Euler equations. Suppose also that the phase $\varphi_0 \in \mathcal{W}^\infty_\infty$ is subjected to (2.1) on the strip $[0,\infty[\times\mathbb{R}^d$ and that it is a global solution of the eiconal equation (ei). Then the Theorem 2.1 can be applied with any $T \in \mathbb{R}^+_*$. It means that no blow up occurs at the level of the equations yielding the profiles U_k, P_k and the phases φ_k. Yet, non linear effects are present. $\triangle$

Remark 2.2.4: The characteristic curves of the field $\partial_t + \mathbf{u}_0 \cdot \nabla_x$ are obtained by solving the differential equation

$$\partial_t \, \Gamma(t,x) \; = \; \mathbf{u}_0\big(t, \Gamma(t,x)\big) \, , \qquad \Gamma(0,x) = x \, .$$

Suppose that the oscillations of the profiles U^*_{k0} are concentrated in some domain $D \subset \mathbb{R}^d$. In other words

$$\text{supp}_x \, U^*_{k0} \; \subset \; D \, , \qquad \forall \, k \in \{1, \cdots, N\} \, .$$

The BKW analysis reveals that for all $t \in [0,T]$ we have

$$\text{supp}_x \, U^*_k(t,\cdot) \; \subset \; \big\{ \, \Gamma(t,x) \, ; \; x \in D \, \big\} \, , \qquad \forall \, k \in \{1, \cdots, N\} \, .$$

The phenomena under study have a finite speed of propagation. $\triangle$

Remark 2.2.5: The influence of dissipation terms will be taken into account in the subsection 4.1. The viscosity we will incorporate is anisotropic. It is small enough in the direction $\nabla \varphi^\varepsilon_\flat$ in order to be compatible with the propagation of oscillations. $\triangle$

2.3 End of the proof of Theorem 2.1.

The Theorem 2.1 is a consequence of the Proposition 4.1 which will be stated and demonstrated in the subsection 4.2. Below, we just explain how to deduce the Theorem 2.1 from the Proposition 4.1 applied with $\nu = 0$.

• **Dictionary between the profiles**. Select arbitrary initial data for

$$\Pi_0(0,x) \, \tilde{U}^*_k(0,x,\theta) \in H^\infty \, , \qquad \langle \tilde{U}_k \rangle (0,x) \in H^\infty \, , \qquad 1 \leq k \leq N \, ,$$

and arbitrary initial data for

$$\varphi_k(0,x) \in H^\infty \, , \qquad 1 \leq k \leq l-1 \, .$$

On the contrary, impose

$$(2.5) \qquad \varphi_k(0,\cdot) \equiv 0\,, \qquad \forall\, k \in \{l,\cdots,N\}\,.$$

The Proposition 4.1 provides with finite sequences

$$\{\tilde{U}_k\}_{1\leq k\leq N}\,, \qquad \{\tilde{P}_k\}_{1\leq k\leq N}\,, \qquad \{\varphi_k\}_{1\leq k\leq N}\,,$$

and source terms

$$\tilde{f}^\varepsilon_\flat(t,x,\theta) \in \mathcal{W}^\infty_T\,, \qquad \tilde{g}^\varepsilon_\flat(t,x,\theta) \in \mathcal{W}^\infty_T\,.$$

such that the associated oscillations

$$\begin{aligned}
\tilde{\mathbf{u}}^\varepsilon_\flat(t,x) &:= \mathbf{u}_0(t,x) + \textstyle\sum_{k=1}^N \varepsilon^{\frac{k}{l}}\ \tilde{U}_k\big(t,x,\varepsilon^{-1}\,\varphi^\varepsilon_\flat(t,x)\big)\,,\\
\tilde{\mathbf{p}}^\varepsilon_\flat(t,x) &:= \mathbf{p}_0(t,x) + \textstyle\sum_{k=1}^N \varepsilon^{\frac{k}{l}}\ \tilde{P}_k\big(t,x,\varepsilon^{-1}\,\varphi^\varepsilon_\flat(t,x)\big)\,,\\
\tilde{\mathbf{f}}^\varepsilon_\flat(t,x) &:= \varepsilon^{-1}\ \tilde{f}^\varepsilon_\flat\big(t,x,\varepsilon^{-1}\,\varphi^\varepsilon_\flat(t,x)\big)\,,\\
\tilde{\mathbf{g}}^\varepsilon_\flat(t,x) &:= \varepsilon^{-1}\ \tilde{g}^\varepsilon_\flat\big(t,x,\varepsilon^{-1}\,\varphi^\varepsilon_\flat(t,x)\big)\,,
\end{aligned}$$

are subjected to

$$\partial_t\tilde{\mathbf{u}}^\varepsilon_\flat+(\tilde{\mathbf{u}}^\varepsilon_\flat\cdot\nabla)\tilde{\mathbf{u}}^\varepsilon_\flat+\nabla\tilde{\mathbf{p}}^\varepsilon_\flat = \tilde{\mathbf{f}}^\varepsilon_\flat = \bigcirc(\varepsilon^{\frac{N+1}{l}-1})\,, \qquad \operatorname{div}\tilde{\mathbf{u}}^\varepsilon_\flat = \tilde{\mathbf{g}}^\varepsilon_\flat = \bigcirc(\varepsilon^{\frac{N+1}{l}-1})\,.$$

The oscillations $\tilde{\mathbf{u}}^\varepsilon_\flat$ and $\tilde{\mathbf{p}}^\varepsilon_\flat$ involve the *complete* phase $\varphi^\varepsilon_\flat(t,x)$ which is the sum of the geometrical phase $\varphi^\varepsilon_g(t,x)$ plus some *adjusting* phase $\varepsilon\,\varphi^\varepsilon_a(t,x)$. More precisely

$$\varphi^\varepsilon_\flat(t,x) := \varphi^\varepsilon_g(t,x) + \varepsilon\ \varphi^\varepsilon_a(t,x)\,, \qquad \varphi^\varepsilon_a(t,x) := \textstyle\sum_{k=l}^N \varepsilon^{\frac{k}{l}-1}\ \varphi_k(t,x)\,.$$

The functions $\tilde{\mathbf{u}}^\varepsilon_\flat$ and $\tilde{\mathbf{p}}^\varepsilon_\flat$ can also be written in terms of the phase φ^ε_g. Indeed, there is a unique decomposition

$$\tilde{\mathbf{u}}^\varepsilon_\flat = \mathbf{u}^\varepsilon_\flat + \mathbf{r}\mathbf{u}^\varepsilon_\flat = \mathbf{u}^\varepsilon_\flat + \bigcirc(\varepsilon^{\frac{N+1}{l}})\,, \qquad \tilde{\mathbf{p}}^\varepsilon_\flat = \mathbf{p}^\varepsilon_\flat + \mathbf{r}\mathbf{p}^\varepsilon_\flat = \mathbf{p}^\varepsilon_\flat + \bigcirc(\varepsilon^{\frac{N+1}{l}})\,,$$

involving the representations

$$(2.6) \qquad \mathbf{u}^\varepsilon_\flat(t,x) = u^\varepsilon_\flat\big(t,x,\varepsilon^{-1}\,\varphi^\varepsilon_g(t,x)\big)\,, \qquad \mathbf{p}^\varepsilon_\flat(t,x) = p^\varepsilon_\flat\big(t,x,\varepsilon^{-1}\,\varphi^\varepsilon_g(t,x)\big)$$

where the profiles $u^\varepsilon_\flat(t,x,\theta)$ and $p^\varepsilon_\flat(t,x,\theta)$ have the form

$$\begin{aligned}
u^\varepsilon_\flat(t,x,\theta) &= \mathbf{u}_0(t,x) + \textstyle\sum_{k=1}^N \varepsilon^{\frac{k}{l}}\ U_k(t,x,\theta)\,,\\
p^\varepsilon_\flat(t,x,\theta) &= \mathbf{p}_0(t,x) + \textstyle\sum_{k=1}^N \varepsilon^{\frac{k}{l}}\ P_k(t,x,\theta)\,.
\end{aligned}$$

The transition from $\tilde{\mathbf{u}}^\varepsilon_\flat$ to $\mathbf{u}^\varepsilon_\flat$ is achieved through the phase shift φ^ε_a

$$\tilde{U}_k(t,x,\varepsilon^{-1}\,\varphi^\varepsilon_\flat) = \tilde{U}_k\big(t,x,\varepsilon^{-1}\,\varphi^\varepsilon_g + \varphi_l + \textstyle\sum_{k=l+1}^N \varepsilon^{\frac{k}{l}-1}\ \varphi_k\big)\,.$$

Use the Taylor formula in order to absorb the small term in the right. It furnishes the following explicit link between the (U_k, P_k) and the $(\tilde{U}_k, \tilde{P}_k)$

$$
\begin{aligned}
U_k\big(t,x,\theta-\varphi_l(t,x)\big) \;&:=\; \tilde{U}_k(t,x,\theta) \,+\, \mathcal{G}^k(\tilde{U}_1,\cdots,\tilde{U}_{k-1})(t,x,\theta)\,,\\
P_k\big(t,x,\theta-\varphi_l(t,x)\big) \;&:=\; \tilde{P}_k(t,x,\theta) \,+\, \mathcal{G}^k(\tilde{P}_1,\cdots,\tilde{P}_{k-1})(t,x,\theta)\,.
\end{aligned} \tag{2.7}
$$

The application $\mathcal{G}^k$ can be put in the form

$$\mathcal{G}^k(\tilde{U}_1,\cdots,\tilde{U}_{k-1}) \;:=\; \textstyle\sum_{p=1}^{k-1} \partial_\theta^p \mathcal{G}_p^k(\tilde{U}_1,\cdots,\tilde{U}_{k-p})\,, \qquad k\in\{1,\cdots,N\}\,.$$

The terms $\mathcal{G}_p^k$ are given by

$$\mathcal{G}_p^k(\tilde{U}_1,\cdots,\tilde{U}_{k-p}) \;:=\; \tfrac{1}{p!}\,\textstyle\sum_{\alpha\in\mathcal{J}_p^k}\ \varphi_{l+1+\alpha_1}\times\cdots\times\varphi_{l+1+\alpha_p}\ \tilde{U}_{\alpha_{p+1}}\,,$$

where the sum is taken over the set

$$
\begin{aligned}
\mathcal{J}_p^k \;:=\; \big\{\,\alpha=(\alpha_1,\cdots,\alpha_p,\alpha_{p+1})\in\mathbb{N}^{p+1}\,;\\
&0\le\alpha_j\le N-l-1\,, \qquad \forall\, j\in\{1,\cdots,p\}\,,\\
&1\le\alpha_{p+1}\le k-p\,, \qquad \alpha_1+\cdots+\alpha_p+\alpha_{p+1}=k-p\,\big\}\,.
\end{aligned}
$$

The relation (2.7) and the definition of $\mathcal{G}^k$ imply that

$$\bar{U}_k(t,x) \;=\; \langle\tilde{U}_k\rangle(t,x)\,, \qquad \forall\, k\in\{1,\cdots,N\}\,, \qquad \forall\, t\in[0,T]\,.$$

Therefore, prescribing the initial data for the $\bar{U}_k$ or the $\langle\tilde{U}_k\rangle$ amounts to the same thing. The condition (2.5) yields

$$\mathcal{G}_p^k(\tilde{U}_1,\cdots,\tilde{U}_{k-p})(0,x,\theta)=0\,, \qquad \forall\, k\in\{1,\cdots,N\}\,.$$

Since $\varphi_l(0,\cdot)\equiv 0$, we have

$$\Pi_0(0,x)\,U_k^*(0,x,\theta) \;=\; \Pi_0(0,x)\,\tilde{U}_k^*(0,x,\theta)\,, \qquad \forall\, k\in\{1,\cdots,N\}\,.$$

It is clearly equivalent to specify the initial data for the $\Pi_0\,U_k^*$ or the $\Pi_0\,\tilde{U}_k^*$.

• **The divergence free relation in the variables (t,x).** Consider the application

$$\mathrm{div}\;:\;H^\infty\;\longrightarrow\;\mathrm{Im}\,(\mathrm{div})\;\subset\;\big\{\,\mathbf{g}\in H^\infty\,;\ \hat{\mathbf{g}}(0)=0\,\big\}\,.$$

We can select some special right inverse.

Lemma 2.1. *There is a linear operator* ridiv $:\mathrm{Im}\,(\mathrm{div})\longrightarrow H^\infty$ *with*

$$\mathrm{div}\circ\mathrm{ridiv}\;\mathbf{g}\;=\;\mathbf{g}\,, \qquad \forall\,\mathbf{g}\in\mathrm{Im}\,(\mathrm{div})\,. \tag{2.8}$$

For all $\iota>0$ and for all $m\in\mathbb{N}$, there is a constant $C_m^\iota>0$ such that

$$\|\,\mathrm{ridiv}\;\mathbf{g}\,\|_{H^m}\le\;C_m\;\;\|\,\mathbf{g}\,\|_{H^{m+1+\frac{d}{2}+\iota}}\,, \qquad \forall\,\mathbf{g}\in\mathrm{Im}\,(\mathrm{div})\,. \tag{2.9}$$

Proof of the Lemma 2.1. Introduce a cut-off function $\psi \in C^\infty(\mathbb{R}^d)$ such that

$$\big\{\,\xi\,;\ \psi(\xi) \neq 0\,\big\} \subset B(0,2]\,, \qquad \big\{\,\xi\,;\ \psi(\xi) = 1\,\big\} \supset B(0,1]\,.$$

For $g \in \mathrm{Im}\,(\mathrm{div})$, take the explicit formula

$$\mathrm{ridiv}\,(\mathbf{g}) := \mathcal{F}^{-1}\,\big(\int_0^1 \nabla_\xi(\psi\,\hat{\mathbf{g}})(r\,\xi)\ dr\ +\ |\xi|^{-2}\ (1-\psi)(\xi)\ \hat{\mathbf{g}}(\xi) \times \xi\,\big)\,.$$

Since $\hat{\mathbf{g}}(0) = 0$, the relation (2.8) is satisfied. For $s > \frac{d}{2}$, the injection $H^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ is continuous. It leads to (2.9). $\Diamond$

• **The Leray projector in the variables** (t,x)**.** Note $\Pi(\xi)$ the orthogonal projector from $\mathbb{R}^d$ onto the plane

$$\xi^\perp\ :=\ \{\,u \in \mathbb{R}^d\,;\ u\cdot\xi = 0\,\}\,.$$

Introduce the closed subspace

$$\mathrm{F}\ :=\ \big\{\,\mathbf{u} \in L^2\,;\ \mathrm{div}\,\mathbf{u} = 0\,\big\} \subset L^2\,.$$

Call P the orthogonal projector from L^2 onto F. It corresponds to the Fourier multiplier

$$P\,\mathbf{u}\ =\ \Pi(D_x)\,\mathbf{u}\ :=\ (2\,\pi)^{-\frac{d}{2}}\ \int_{\mathbb{R}^d} e^{i\,x\cdot\xi}\ \Pi(\xi)\ \hat{\mathbf{u}}(\xi)\ d\xi\,.$$

The application P is the Leray projector onto the space of divergence free vector fields. It is a self-adjoint operator such that

$$\ker\,\mathrm{div}\ =\ \mathrm{Im}\,P\,, \qquad \mathrm{Im}\,\nabla\ =\ \big(\ker\,(\mathrm{div})\big)^\perp\ =\ \ker\,P\,.$$

Consider the Cauchy problem

$$\partial_t\mathbf{u} + \nabla\mathbf{p} = \mathbf{f}\,, \qquad \mathrm{div}\,\mathbf{u} = 0\,, \qquad \mathbf{u}(0,\cdot) = \mathbf{h}$$

with data $\mathbf{f} \in L^2_T$ and $\mathbf{h} \in L^2$. It leads to the equivalent conditions

$$\partial_t\mathbf{u} = P\,\mathbf{f}\,, \qquad \mathbf{u}(0,\cdot) = P\,\mathbf{h}\,, \qquad \nabla\mathbf{p} = (\mathrm{Id} - P)\,\mathbf{f}\,.$$

Now we come back to the proof of Theorem 2.1. It remains to absorb the term $\tilde{\mathbf{g}}^\varepsilon_\flat \in \mathrm{Im}\,(\mathrm{div})$. To this end, take $\iota = \frac{1}{2\,l}$. Define $\mathbf{u}^\varepsilon_\flat$ and $\mathbf{p}^\varepsilon_\flat$ as in (2.2) with the U_k and P_k of (2.7). Introduce

$$\mathbf{cu}^\varepsilon_\flat\ :=\ \mathbf{ru}^\varepsilon_\flat - \mathrm{ridiv}\,\tilde{\mathbf{g}}^\varepsilon_\flat = \bigcirc(\varepsilon^{\frac{N}{l}-2-\frac{d}{2}})\,, \qquad \mathbf{cp}^\varepsilon_\flat\ :=\ \mathbf{rp}^\varepsilon_\flat = \bigcirc(\varepsilon^{\frac{N+1}{l}})\,.$$

After substitution in $(\mathcal{E})$, we lose again a power of ε. We find

$$\begin{aligned}\mathbf{f}^\varepsilon_\flat = \tilde{\mathbf{f}}^\varepsilon_\flat &- (\mathrm{ridiv}\,\tilde{\mathbf{g}}^\varepsilon_\flat\cdot\nabla)\,\tilde{\mathbf{u}}^\varepsilon_\flat - (\tilde{\mathbf{u}}^\varepsilon_\flat\cdot\nabla)\,\mathrm{ridiv}\,\tilde{\mathbf{g}}^\varepsilon_\flat \\ &- \partial_t\mathrm{ridiv}\,\tilde{\mathbf{g}}^\varepsilon_\flat + (\mathrm{ridiv}\,\tilde{\mathbf{g}}^\varepsilon_\flat\cdot\nabla)\,\mathrm{ridiv}\,\tilde{\mathbf{g}}^\varepsilon_\flat\ =\ \bigcirc(\varepsilon^{\frac{N}{l}-3-\frac{d}{2}})\,.\end{aligned}$$

The Theorem 2.1 looks like classical statements in one phase non linear geometric optics except that the phase φ^ε_g does depend on ε. In the next chapter, we examine the part of the φ_k which make up φ^ε_g and φ^ε_a.

3 The cascade of phases.

Turbulence and *intermittency* are topics which represent extremely different points of view. Two approaches compete:

a) The deterministic approach which study the time evolution of flows arising in fluid mechanics [1]-[3]-[11]-[12]-[24].

b) The statistical approach in which the velocity of the fluid is a random variable [14]-[22].

The Theorem 2.1 is mainly connected with a). It brings various informations related to the propagation of quasi-singularities. These aspects are detailed at first. Then we briefly explain b) and we draw (in the setting of the Theorem 2.1) a phenomenological comparison between a) and b).

3.1 Microstructures.

The result 2.1 is concerned with the convection of microstructures. It is linked with the multiple scale approach of [24] and [3]. In [24] the authors look for BKW solutions $\mathbf{u}^\varepsilon_a$ in the form

$$\mathbf{u}^\varepsilon_a(t,x) \, = \, \mathbf{u}_0(t,x) \, + \, U^*_0\big(t,x,\varepsilon^{-1}\,t,\varepsilon^{-1}\,\vec{\varphi}_0(t,x)\big) \, + \, \bigcirc(\varepsilon)\,.$$

In the more recent paper [3], the selected expansion is

$$\mathfrak{u}^\varepsilon_a(t,x) \, = \, \mathbf{u}_0(t,x) \, + \, \varepsilon^{\frac{1}{3}} \, U_1\big(t,x,\varepsilon^{-\frac{2}{3}}\,t,\varepsilon^{-1}\,\vec{\varphi}_0(t,x)\big) \, + \, \bigcirc(\varepsilon^{\frac{2}{3}})\,.$$

Both articles [3] and [24] use homogenization techniques. They perform computations involving expressions as $\mathbf{u}^\varepsilon_a$ or $\mathfrak{u}^\varepsilon_a$. Simplifications (supported by engineering experiments) are made in order to get effective equations for the evolution of $(\mathbf{u}_0,U^*_0)$ or $(\mathbf{u}_0,U_1)$.

Consider the simple case of one phase expansions (that is when $\vec{\varphi}_0 \equiv \varphi_0$ is a scalar valued function). Reasons why a complete mathematical analysis based on $\mathbf{u}^\varepsilon_a$ or $\mathfrak{u}^\varepsilon_a$ is not available can be drawn from the Theorem 2.1. For instance, look at $\mathfrak{u}^\varepsilon_a$. When $l=3$, the oscillation $\mathfrak{u}^\varepsilon_a$ involves the same scales as $\mathbf{u}^\varepsilon_{(3,N)}$ since

$$\varepsilon^{-1}\,\varphi^\varepsilon_g(t,x) \, = \, \varepsilon^{-1}\,\varphi_0(t,x) + \varepsilon^{-\frac{2}{3}}\,\varphi_1(t,x) + \varepsilon^{-\frac{1}{3}}\,\varphi_2(t,x)\,.$$

Now the analogy stops here since in general $\varphi_1(t,x) \not\equiv t$ and $\varphi_2(t,x) \not\equiv 0$. These are geometrical obstructions which prevent to describe the propagation by way of $\mathfrak{u}^\varepsilon_a$. The asymptotic expansion $\mathfrak{u}^\varepsilon_a$ is not suitable.

Analogous arguments concerning $\mathbf{u}^\varepsilon_a$ will be presented in the paragraph 3.5.

3.2 The geometrical phase.

Let us examine more carefully how the expression φ_g^ε is built. Because of the condition (2.1), for ε small enough, it is still not stationary

$$(3.1) \qquad \exists\, \varepsilon_0 > 0\,; \qquad \nabla\varphi_g^\varepsilon(t,x) \neq 0\,, \qquad \forall\, (\varepsilon,t,x) \in]0,\varepsilon_0] \times [0,T] \times \mathbb{R}^d\,.$$

In fact, the function φ_g^ε comes from the approximate eiconal equation

$$\partial_t \varphi_g^\varepsilon + (\bar{u}_\flat^\varepsilon \cdot \nabla)\varphi_g^\varepsilon = \bigcirc(\varepsilon)$$

which is equivalent to

$$\partial_t \varphi_k + \mathbf{u}_0 \cdot \nabla \varphi_k + \textstyle\sum_{j=0}^{k-1} \bar{U}_{k-j} \cdot \nabla\varphi_j = 0\,, \qquad \forall\, k \in \{1,\cdots,l-1\}\,.$$

The family $\{\mathbf{u}_\flat^\varepsilon(t,x)\}_{\varepsilon \in\,]0,1]}$ has an $\varepsilon-$stratified regularity [19] with respect to the phase φ_g^ε. This is a *geometrical* information.

3.3 Closure problems.

We have explained why appealing only to φ_0 is not sufficient. It turns out that BKW computations relying only on the geometrical phase φ_g^ε come also to nothing. This is a subtle aspect when proving the Theorem 2.1. We lay now stress on it.

For all $N \in \mathbb{N}_*$, the application $\mathcal{G}$ defined below is one to one

$$\mathcal{G} \;:\; (\mathcal{W}_T^\infty)^N \quad \longrightarrow \quad (\mathcal{W}_T^\infty)^N$$

$$\begin{pmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ \vdots \\ \tilde{U}_N \end{pmatrix} (t,x,\theta) \quad \longmapsto \quad \begin{pmatrix} \tilde{U}_1 \\ \tilde{U}_2 + \mathcal{G}^1(\tilde{U}_1) \\ \vdots \\ \tilde{U}_N + \mathcal{G}^N(\tilde{U}_1,\cdots,\tilde{U}_{N-1}) \end{pmatrix} (t,x,\theta+\varphi_l(t,x))\,.$$

Once the U_j or the $\tilde{U}_j$ are known, it is entirely equivalent to use $\mathbf{u}_\flat^\varepsilon$ or $\tilde{\mathbf{u}}_\flat^\varepsilon$. Before the U_j or the $\tilde{U}_j$ have been identified, in particular when performing the BKW calculus, it is deeply different to employ $\mathbf{u}_\flat^\varepsilon$ or $\tilde{\mathbf{u}}_\flat^\varepsilon$. Indeed, there is a unique choice of the φ_k with $l \leq k \leq N$, which imposes a *specific hierarchy* between the profiles $\tilde{U}_k$, which makes possible the *triangulation* of the equations obtained by the formal computations.

In the subsection 2.3, we will perform the BKW analysis with the profiles $\tilde{U}_k$. It yields a sequence of equations

$$(3.2) \qquad \tilde{X}^k(\tilde{U}_1,\cdots,\tilde{U}_{k+l}) \;=\; 0\,, \qquad 1 \leq k \leq N\,.$$

As usual in non linear geometric optics, this can be rewritten in order to find a sequence of well-posed equations

$$\dot{X}^k(\dot{U}_k) \;=\; \mathcal{F}(\dot{U}_1,\cdots,\dot{U}_{k-1})\,, \qquad 1 \leq k \leq N\,, \tag{3.3}$$

where the $\dot{U}_k$ are made of pieces of the $\tilde{U}_j$. Of course, the equation (3.3) can be interpreted in terms of the $\tilde{U}_j$ and then in terms of the U_j. In this second step, it requires to implement the phase shift φ_l and the transformations $\mathcal{G}_p^j$ with $1 \leq j \leq k-1$ and $1 \leq p \leq j$. Now, the BKW analysis reveals that φ_l or the various coefficients φ_i which appear in the definition of such $\mathcal{G}_p^j$ do not depend only on $(\dot{U}_1,\cdots,\dot{U}_k)$ but also on some $\dot{U}_i$ with $i > k$. The resulting system is therefore underdetermined. Computations involving the functions U_j lead to a sequence of equations which are not closed.

The insertion of the phases φ_k with $1 \leq k \leq N$ is an elegant way to introduce $\mathcal{G}$. The change of variables $\mathcal{G}$, though it is a function of $(U_1,\cdots,U_N)$, is needed to progress. It allows to get round *closure problems.*

3.4 Compensated compactness.

Dissipation terms can be incorporated in the discussion. In the variables (t,x), the addition of some viscosity κ is compatible with the propagation of oscillations if for instance $\kappa = \nu\,\varepsilon^2$. There are approximate solutions $(\mathbf{u}^\varepsilon_\flat,\mathbf{p}^\varepsilon_\flat)$ of the Navier-Stokes equations. They satisfy (2.2) and

$$\partial_t \mathbf{u}^\varepsilon_\flat + (\mathbf{u}^\varepsilon_\flat\cdot\nabla)\mathbf{u}^\varepsilon_\flat + \nabla\mathbf{p}^\varepsilon_\flat = \nu\,\varepsilon^2\;\Delta_x \mathbf{u}^\varepsilon_\flat + \mathbf{f}^\varepsilon_\flat\,, \qquad \text{div}\;\mathbf{u}^\varepsilon_\flat = 0\,,$$

with $\mathbf{f}^\varepsilon_\flat = \bigcirc(\varepsilon^\infty)$. When $\nu > 0$, Leray's theorem provides with global weak solutions $(\mathbf{u}^\varepsilon,\mathbf{p}^\varepsilon)(t,x)$ of the following Cauchy problem

$$\left\{ \begin{array}{l} \partial_t \mathbf{u}^\varepsilon + (\mathbf{u}^\varepsilon\cdot\nabla)\mathbf{u}^\varepsilon + \nabla\mathbf{p}^\varepsilon = \nu\,\varepsilon^2\;\Delta_x \mathbf{u}^\varepsilon\,, \qquad \text{div}\;\mathbf{u}^\varepsilon = 0\,, \\ \mathbf{u}^\varepsilon(0,\cdot) \equiv \mathbf{u}^\varepsilon_\flat(0,\cdot)\,. \end{array} \right.$$

Suppose now that $\mathbf{u}_0 \equiv 0$. Then, we have also the uniform controls

$$\begin{array}{l} \sup\,\big\{\,\|\,\varepsilon^{-\frac{1}{l}}\,\mathbf{u}^\varepsilon\,\|_{L^2_T}\,;\;\varepsilon\in]0,1]\,\big\}\;\leq\; C\;<\;\infty\,, \\ \sup\,\big\{\,\nu\;\varepsilon^2\;\int_0^T\;\|\,\varepsilon^{-\frac{1}{l}}\,\mathbf{u}^\varepsilon(t,\cdot)\,\|^2_{H^1(\mathbb{R}^d)}\;\,dt\,;\;\varepsilon\in]0,1]\,\big\}\;\leq\; C\;<\;\infty\,. \end{array} \tag{3.4}$$

Arguments issued from the theory of compensated compactness [16] can be employed to study the sequence $\{\varepsilon^{-\frac{1}{l}}\,\mathbf{u}^\varepsilon\}_\varepsilon$. In the spirit of [11] or [12], we can try to exploit the informations contained in (3.4) and the equation on $\mathbf{u}^\varepsilon$ in order to describe the asymptotic behaviour when ε goes to zero of the functions $\varepsilon^{-\frac{1}{l}}\,\mathbf{u}^\varepsilon$. However this approach seems to be not applicable here.

Indeed, *obvious* instabilities occur. The related mechanisms, which induce the non linear instability of Euler equations, are detailed in the paragraph 5.1. Below, we just give an intuitive idea of what can happen. Use the representation $\tilde{\mathbf{u}}_\flat^\varepsilon$ involving the phase $\varphi_\flat^\varepsilon$. The determination of the intermediate term φ_l requires to identify $\langle \tilde{U}_l \rangle$ and $\tilde{U}_{l-1}^*$. This is a consequence of the equations (4.18) and (4.21).

In view of the formula (2.7), when φ_l is modified by an amount of $\delta\varphi_l$, the quantity $U_1(t,x,\theta)$ undergoes a perturbation of the same order $\delta\varphi_l$. When dealing with quasi-singularities, some quantities with ε in factor (like $\langle \tilde{U}_l \rangle$) or with $\varepsilon^{1-\frac{1}{l}}$ in factor (like $\tilde{U}_{l-1}^*$) can control informations of size $\varepsilon^{\frac{1}{l}}$. This fact is expressed by the following rules of transformation

$$
(3.5) \qquad \begin{array}{lcl} \langle \tilde{U}_l \rangle \ / \ \langle \tilde{U}_l \rangle \, + \, \delta \langle \tilde{U}_l \rangle & \Longrightarrow & \mathbf{u}_\flat^\varepsilon \ / \ \mathbf{u}_\flat^\varepsilon + \bigcirc(\varepsilon^{\frac{1}{l}}) \ \delta \langle \tilde{U}_l \rangle \,, \\ \tilde{U}_{l-1}^* \ / \ \tilde{U}_{l-1}^* \, + \, \delta \tilde{U}_{l-1}^* & \Longrightarrow & \mathbf{u}_\flat^\varepsilon \ / \ \mathbf{u}_\flat^\varepsilon + \bigcirc(\varepsilon^{\frac{1}{l}}) \ \delta \tilde{U}_l^* \,. \end{array}
$$

Now reverse the preceding reasoning. To describe features in the principal oscillating term $\varepsilon^{\frac{1}{l}} \ U_1^*\big(t,x,\varepsilon^{-1}\,\varphi_g^\varepsilon(t,x)\big)$, we must identify φ_l which means to obtain $\langle \tilde{U}_l \rangle$ and $\tilde{U}_{l-1}^*$. In other words, we need to know quantities which have respectively ε and $\varepsilon^{1-\frac{1}{l}}$ in factor. When $l \geq 2$ such informations are clearly not reachable by rough controls as (3.4).

This discussion indicates that the study of turbulent regimes requires to combine at least geometrical aspects, multiphase analysis and high order expansions. The tools of non linear geometric optics seem to be appropriate. Some attempts in this direction have already been made.

3.5 Non linear geometric optics.

We make in this paragraph 3.5 several comments about non linear geometric optics. They concern both old [18]-[19]-[26] and recent [6]-[7]-[8] results which all are devoted to one phase expansions of the type

$$
(3.6) \qquad \mathbf{u}_\natural^\varepsilon(t,x) \, := \, \mathbf{u}_0(t,x) \, + \, \textstyle\sum_{k=1}^\infty \varepsilon^{\frac{k}{l}} \ U_k\big(t,x,\varepsilon^{-1}\,\varphi_0(t,x)\big) \,.
$$

When $l = 1$, one is faced with *weakly non linear geometric optics.* The asymptotic behavior and the stability of $\mathbf{u}_\natural^\varepsilon$ are well understood. In fact a complete theory has been achieved in the general framework of multidimensional systems of conservation laws (see [18]-[19] and the related references). Because of the formation of shocks, the life span of exact solutions close to $\mathbf{u}_\natural^\varepsilon$ does not go beyond $T \simeq 1$.

When $l = 2$, expressions as $\mathbf{u}^\varepsilon_\natural$ are called *strong* oscillations. The construction of such BKW solutions can be undertaken only if the system of conservation laws has a special structure. *Transparency* conditions are needed to progress. They can be deduced from the presence of a linearly degenerate field [7]. In the hyperbolic situation the family $\{\mathbf{u}^\varepsilon_\natural\}_{\varepsilon \in]0,1]}$ is unstable [7] on the interval $[0, T]$. It becomes stable on condition that a small viscosity is incorporated [6]. Applications can be given to describe large-scale motions in the atmosphere [6].

Compressible Euler equations are the prototype of a non linear hyperbolic system having a linearly degenerate field. After a finite time, singularities appear. These correspond to the generation of shocks by compression [27]. The situation is different in the incompressible setting. There is no genuine shock and the production of singularities poses a much more subtle problem [2]-[9] which up to now remains basically open.

Incompressible fluid equations lie at an extreme end in the sense that they are the most *degenerate* (or the most *linear*) equations which have just been mentioned. Following the approach of [20] related to *transparency*, repeating the reasoning which goes from [18]-[19] to [6]-[7], one expects to go further than $l = 2$ when dealing with $(\mathcal{E})$. Now, this is precisely what says the Theorem 2.1 since it allows to reach any $l \in \mathbb{N}_*$!

To tackle the limit case $l = \infty$, one is tempted to look at asymptotic expansions of the form

$$(3.7) \qquad \mathbf{u}^\varepsilon_\infty(t,x) := \textstyle\sum_{k=0}^{\infty} \varepsilon^k \, U_k\big(t,x,\varepsilon^{-1}\,\varphi_0(t,x)\big)\,, \qquad \partial_\theta U_0^* \not\equiv 0\,.$$

The oscillations contained in $\mathbf{u}^\varepsilon_\infty$ have a *large* amplitude. Modulation equations for U_0 are proposed in [26]. However these transport equations are not hyperbolic so that they are ill posed (in the sense of Hadamard) with respect to the initial value problem. It confirms that a BKW construction based on (3.7) is not relevant[2].

The contribution [26] does not explain why the expansion (3.7) is not the good one. We come back below to this point. At first sight the Theorem 2.1 does not include large amplitude waves since $\mathbf{u}^\varepsilon_\flat - \mathbf{u}_0 = \bigcirc(\varepsilon^{\frac{1}{l}}) \ll \bigcirc(1)$. A change of variables leads to recant this impression. Suppose that $\mathbf{u}_0 \equiv 0$ and $\partial_\theta U_1^* \not\equiv 0$. Then define

$$\dot{\mathbf{u}}^\varepsilon_\flat(t,x) := \varepsilon^{-\frac{1}{l}}\ \mathbf{u}^\varepsilon_\flat(\varepsilon^{-\frac{1}{l}}\, t,x)\,, \qquad \dot{\mathbf{p}}^\varepsilon_\flat(t,x) := \varepsilon^{-\frac{2}{l}}\ \mathbf{p}^\varepsilon_\flat(\varepsilon^{-\frac{1}{l}}\, t,x)\,.$$

[2]The singularities are carried here by the velocity field. The discussion is very different when the oscillations are polarized on the entropy [8].

Observe that the structure of $\dot{\mathbf{u}}^\varepsilon_\flat$ and $\dot{\mathbf{p}}^\varepsilon_\flat$ is very different from the one in (3.7) since we have

$$\begin{aligned}
\dot{\mathbf{u}}^\varepsilon_\flat(t,x) &= \textstyle\sum_{k=1}^\infty \varepsilon^{\frac{k-1}{l}}\; U_k\big(\varepsilon^{-\frac{1}{l}}\, t, x, \varepsilon^{-1}\, \varphi^\varepsilon_g(\varepsilon^{-\frac{1}{l}}\, t, x)\big) + \varepsilon^{-\frac{1}{l}}\; \mathbf{cu}^\varepsilon_\flat(\varepsilon^{-\frac{1}{l}}\, t, x)\,, \\
\dot{\mathbf{p}}^\varepsilon_\flat(t,x) &= \textstyle\sum_{k=1}^\infty \varepsilon^{\frac{k-2}{l}}\; P_k\big(\varepsilon^{-\frac{1}{l}}\, t, x, \varepsilon^{-1}\, \varphi^\varepsilon_g(\varepsilon^{-\frac{1}{l}}\, t, x)\big) + \varepsilon^{-\frac{2}{l}}\; \mathbf{cp}^\varepsilon_\flat(\varepsilon^{-\frac{1}{l}}\, t, x)\,.
\end{aligned}$$

The functions $\dot{\mathbf{u}}^\varepsilon_\flat$ and $\dot{\mathbf{p}}^\varepsilon_\flat$ satisfy

$$\partial_t \dot{\mathbf{u}}^\varepsilon_\flat + (\dot{\mathbf{u}}^\varepsilon_\flat \cdot \nabla)\dot{\mathbf{u}}^\varepsilon_\flat + \nabla \dot{\mathbf{p}}^\varepsilon_\flat = \dot{\mathbf{f}}^\varepsilon_\flat\,, \qquad \operatorname{div} \dot{\mathbf{u}}^\varepsilon_\flat = 0\,, \qquad \dot{\mathbf{f}}^\varepsilon_\flat(t,x) = \varepsilon^{-\frac{2}{l}}\, \mathbf{f}^\varepsilon_\flat(\varepsilon^{-\frac{1}{l}}\, t, x)\,.$$

The functions $\dot{\mathbf{u}}^\varepsilon_\flat$ are oscillations of the order 1. They are approximate solutions of $(\mathcal{E})$ on the *small* interval $[0, \varepsilon^{\frac{1}{l}}\, T]$. Indeed, for all $m \in \mathbb{N}$, the family $\{\dot{\mathbf{f}}^\varepsilon_\flat\}_\varepsilon$ is subjected to the uniform majoration

$$\sup_{\varepsilon \in\,]0,\varepsilon_0]} \quad \varepsilon^{-\frac{N}{l}+\frac{2}{l}+3+m} \;\; \| \, \dot{\mathbf{f}}^\varepsilon_\flat \, \|_{\mathcal{W}^m_{\varepsilon^{(1/l)}\, T}} \, < \, \infty\,.$$

If moreover $N = +\infty$ and

$$\text{(3.8)} \qquad \begin{aligned} &\varphi_1(0,\cdot) \,\equiv\, \cdots \,\equiv\, \varphi_{l-1}(0,\cdot) \,\equiv\, 0\,, \\ &U_{k+1}(0,\cdot) \,\equiv\, 0\,, \qquad \forall\, k \in \mathbb{N} \setminus (l\,\mathbb{N})\,, \end{aligned}$$

the trace $\dot{\mathbf{u}}^\varepsilon_\flat(0,\cdot)$ has the form

$$\dot{\mathbf{u}}^\varepsilon_\flat(0,x) \;=\; \textstyle\sum_{k=0}^\infty \varepsilon^k\; U_{1+l\,k}\big(0, x, \varepsilon^{-1}\, \varphi_0(0,x)\big)\,, \qquad \partial_\theta U_1^* \,\not\equiv\, 0\,.$$

At the time $t = 0$, we recover (3.7). Now the construction underlying the Theorem 2.1 reveals that in general

$$\text{(3.9)} \qquad \varphi_k(t,\cdot) \,\not\equiv\, 0\,, \qquad \forall\, t \in]0,T]\,, \qquad \forall\, k \in \{2, \cdots, l-1\}\,.$$

The functions φ_j with $j \in \{2, \cdots, l-1\}$ are not present when $t = 0$. But the description of $\dot{\mathbf{u}}^\varepsilon_\flat(t,\cdot)$ on the interval $[0, \varepsilon^{1-\frac{k}{l}}\, T]$ with $k \in \{2, \cdots, l-1\}$ requires the introduction of the *phase shifts* φ_j for $j \in \{2, \cdots, k\}$. More generally, the description of $\dot{\mathbf{u}}^\varepsilon_\flat(t,\cdot)$ on the whole interval $[0,T]$ needs the introduction of an *infinite cascade* of phases $\{\varphi_j\}_{j\in\mathbb{N}_*}$.

Such a phenomenon does not occur when constructing large amplitude oscillations for systems of conservation laws in one space dimension [10]-[13]. It is specific to the multidimensional framework. It explains why the classical approach of [26] fails.

It seems that the creation of the φ_j is due to mechanisms which have not already been studied. It is not linked with *resonances*. It is related neither to *dispersive* nor to *diffractive* effects.

Remark 3.5.1 (about φ_1): The term φ_1 does not appear if $\varphi_1(0,\cdot) \equiv 0$ and $\bar{U}_1(0,\cdot) \equiv 0$. When these two conditions are not verified, the phase shift φ_1 can be absorbed by the technical trick exposed in [7]. Just replace $\mathbf{u}_0(0,\cdot)$ by $\mathbf{u}_0(0,\cdot) + \delta\,\bar{U}_1(0,\cdot)$. Perform the BKW calculus with a fixed $\delta > 0$. Then choose $\delta = \varepsilon$. $\triangle$

Remark 3.5.2 (about φ_2): In general, we have $\varphi_2 \not\equiv 0$ even if

$$\varphi_1(0,\cdot) \equiv \varphi_2(0,\cdot) \equiv 0\,, \qquad \bar{U}_1(0,\cdot) \equiv \bar{U}_2(0,\cdot) \equiv 0\,.$$

Indeed the time evolution of $\bar{U}_2$ is governed by (4.22). It involves the source term $\operatorname{div} \langle U_1^* \otimes U_1^* \rangle$ which is able to awake $\bar{U}_2$. This influence can then be transmitted to φ_2 through the transport equation

$$\partial_t \varphi_2 + (\mathbf{u}_0 \cdot \nabla)\varphi_2 + (\bar{U}_1 \cdot \nabla)\varphi_1 + (\bar{U}_2 \cdot \nabla)\varphi_0 \;=\; 0\,. \tag{3.10}$$

Likewise, the other terms $\varphi_3, \cdots, \varphi_{l-1}$ are in general non trivial even if

$$\varphi_1(0,\cdot) \equiv \cdots \equiv \varphi_{l-1}(0,\cdot) \equiv 0\,, \qquad \bar{U}_1(0,\cdot) \equiv \cdots \equiv \bar{U}_{l-1}(0,\cdot) \equiv 0\,.$$

There is no more trick which allows to get rid of $\varphi_2, \cdots, \varphi_{l-1}$. $\triangle$

Remark 3.5.3 (why turbulent flows ?): The introduction of the phase shifts φ_k with $2 \leq k \leq l-1$ cannot be avoided. Therefore the difficulties that we deal with appear from $l = 3$. When $l \geq 3$, the characteristic rate e of eddy dissipation is bigger than one [3]. This is the reason why such situations are refered to *turbulent regimes.* $\triangle$

Remark 3.5.4 (about shear layers): We have said in the introduction that the expression $\mathbf{u}_s^\varepsilon$ given by formula (1.1) is of a very special form. Let us explain why. Change the variable t into $\varepsilon^{\frac{1}{l}}\, t$ and $\mathbf{u}_s^\varepsilon$ into $\dot{\mathbf{u}}_s^\varepsilon := \varepsilon^{\frac{1}{l}}\, \mathbf{u}_s^\varepsilon$. The main phase $\varphi_0(t,x) \equiv x_2$ remains the same. Now we are faced with

$$\dot{\mathbf{u}}_s^\varepsilon(t,x) \;:=\; {}^t\big(\varepsilon^{\frac{1}{l}}\, \mathbf{g}(x_2, \varepsilon^{-1}\, x_2), 0, \varepsilon^{\frac{1}{l}}\, \mathbf{h}\big(x_1 - \varepsilon^{\frac{1}{l}}\, \mathbf{g}(x_2, \varepsilon^{-1}\, x_2)\, t, x_2, \varepsilon^{-1}\, x_2\big)\big)\,.$$

It is still a solution of Euler equations. Now it falls in the framework of the Theorem 2.1. The constraints on $\bar{U}_2 = {}^t(\bar{U}_2^1, \bar{U}_2^2, \bar{U}_2^3)$ reduce to

$$\bar{U}_2^1 \;\equiv\; \bar{U}_2^2 \;\equiv\; 0\,, \qquad \partial_t \bar{U}_2^3 \;+\; \langle \mathbf{g}\, \partial_1 \mathbf{h} \rangle \;=\; 0\,.$$

The contribution $\bar{U}_2$ is non trivial but it is polarized so that $\bar{U}_2 \cdot \nabla \varphi_0 \equiv 0$. Therefore it does not produce the phase shift φ_2. The same phenomenon occurs concerning $\varphi_3, \cdots, \varphi_{l-1}$. These terms are not present. It turns out that the expansion $\mathbf{u}_s^\varepsilon$ involves only the phase $\varphi_0(t,x) \equiv x_2$. $\triangle$

The choice for the amplitude of the oscillations is very important. It is strongly related to the scale of time T under consideration. The idea is to increase the time of propagation T to reach the regime where non linear effects appear. Starting with some large amplitude high frequency waves

$$\mathbf{u}^{\varepsilon}_{\infty}(0,x) \;=\; U_0\big(0,x,\varepsilon^{-1}\,\varphi_0(0,x)\big) \;+\; \bigcirc(\varepsilon)\,, \qquad \partial_\theta U_0^*(0,\cdot) \not\equiv 0\,,$$

the preceding discussion can be summarized by the following diagram:

$T \simeq 1$			
	infinite cascade of phases $\varphi_0 - (\varphi_1) - \cdots$	turbulent flows	incompressible fluid equations
$T \simeq \varepsilon^{\frac{1}{3}}$			
	$\varphi_0 - (\varphi_1) - \varphi_2$	turbulent flows	incompressible fluid equations
$T \simeq \varepsilon^{\frac{1}{2}}$			
	$\varphi_0 - (\varphi_1)$	strong oscillations [6] − [7]	systems of conservation laws with a linearly degenerate field
$T \simeq \varepsilon$			
	φ_0	weakly non linear geometric optics [18] − [19]	systems of conservation laws
$T = 0$			
	phases	**regimes**	**equations**

This picture allows to understand the position of the actual paper in comparison with previous results.

3.6 The statistical approach.

It deals mainly with *quantitative* informations obtained at the level of expressions, say $\mathfrak{u}(x)$, which in general do not depend on the time t. The introduction of $\mathfrak{u}$ can be achieved by looking at *stationary statistical* solutions [14] of the Navier-Stokes equations that is

$$\mathfrak{u}(x) \;\equiv\; \lim_{T \longrightarrow \infty} \quad \frac{1}{T} \; \int_0^T \; \mathbf{u}(t,x)\; dt$$

or in conjunction with the *ensemble average operator* ([22]-V-6) marked by the brackets $< \cdot >$. We will follow this second option. The description below is extracted from the book of M. Lesieur [22] (chapters V and VI). We work with $d = 3$. Interesting quantities are the mean kinetic energy

$$\frac{1}{2} \ < \mathfrak{u}(x)^2 > \, \sim \, \int_{\mathbb{R}^3} |\mathfrak{u}(x)|^2 \ dx \, ,$$

the enstrophy (that is the space integral of the square norm of the vorticity)

$$\frac{1}{2} \ < \omega(x)^2 > \, \sim \, \int_{\mathbb{R}^3} |\omega(x)|^2 \ dx \, , \qquad \omega(x) := \nabla \wedge \mathfrak{u}(x)$$

and the rate of dissipation $e \ \sim \ \kappa \ \ < \omega(x)^2 >$. In the setting of *isotropic* turbulence, these quantities can be expressed in terms of a scalar function $k \longmapsto E(k)$. The real number $E(k)$ represents the density of kinetic energy at wave number k (or the kinetic energy in Fourier space integrated on a sphere of radius k). The relations are the following

[22]-V-10-4 $\qquad \frac{1}{2} \ < \mathfrak{u}(x)^2 > \, = \, \int_0^{+\infty} \ E(k) \ dk \, .$

[22]-V-10-15 $\qquad \frac{1}{2} \ < \omega(x)^2 > \, = \, \int_0^{+\infty} \ k^2 \ E(k) \ dk \, .$

[22]-VI-3-15 $\qquad e \ = \ 2 \ \kappa \ \int_0^{+\infty} \ k^2 \ E(k) \ dk \, .$

Kolmogorov's theory assumes that

[22]-VI-4-1 $\qquad \exists \, c > 0 \, ; \qquad E(k) \ = \ c \ e^{2/3} \ k^{-5/3} \, , \qquad \forall \, k \in [k_i, k_d] \, .$

This law is valid up to the frequency k_d with

[22]-VI-4-2 $\qquad k_d \ \sim \ (\, e \, / \, \kappa^3 \,)^{1/4} \, .$

The small quantity $\varepsilon := k_d^{-1}$ is the Kolmogorov dissipative scale. The relations [22]-VI-3-15 and [22]-VI-4-2 imply that the rate of injection of kinetic energy e is linked to the number l according to $e \sim \varepsilon^{-1+\frac{3}{l}}$. We recover here that $e \sim 1$ when $l = 3$ (see [3]).

A starting point for the conventional theory of turbulence is the notion that, on average, kinetic energy is transfered from low wave numbers modes to high wave numbers modes. A recent paper [14] put forward the following idea: in the spectral region below that of injection of energy, an inverse (from high to low modes) transfer of energy takes place. At any rate, it is a central question to determine how the kinetic energy is distributed.

3.7 Phenomenological comparison.

The statistical approach is concerned with the spectral properties of solutions. Below, we draw a parallel with the propagation of quasi-singularities as it is described in the Theorem 2.1.

Let us examine how the square $\mathcal{F}(\mathbf{u}^\varepsilon_\flat)(t,\xi)^2$ of the Fourier transform of $\mathbf{u}^\varepsilon_\flat(t,x)$ is distributed. To this end, consider the application

$$\begin{array}{rcl} \tilde{E}(t,\cdot)\,:\,\mathbb{R}^+ & \longrightarrow & \mathbb{R}^+ \\ k & \longmapsto & \tilde{E}(t,k)\,:=\,\int_{\{\xi\in\mathbb{R}^d\,;\,|\xi|=k\}}\ |\mathcal{F}(\mathbf{u}^\varepsilon_\flat)(t,\xi)|^2\ d\sigma(\xi)\,. \end{array}$$

The initial data $\mathbf{u}^\varepsilon_\flat(0,\cdot)$ has a *spectral gap*. In another words, the graph of the function $k \longmapsto \tilde{E}(0,k)$ appears concentrated around the two characteristic wave numbers $k \simeq 1$ and $k \simeq \varepsilon^{-1} = k_d$. In view of (3.9), this situation does not persist. At the time $t = \varepsilon^{\frac{1}{l}}$, the concentration is around l characteristic wave numbers which are intermediate between the two preceding ones. This corresponds to a *discrete* cascade of energy.

Suppose now (3.8) and consider $\dot{\mathbf{u}}^\varepsilon_\flat$. The life span of $\dot{\mathbf{u}}^\varepsilon_\flat(t,\cdot)$ is $\varepsilon^{\frac{1}{l}}\,T$. There are various manners to get a family $\{\dot{\mathbf{u}}^\varepsilon_\flat(t,\cdot)\}_{\varepsilon\in\,]0,1]}$ which is defined on some interval $[0,\tilde{T}]$ with $\tilde{T}>0$ independent on ε. In particular, we can

a) Select any $\tilde{T}>0$ when $T=+\infty$. However nothing guarantees that the functions $\dot{\mathbf{u}}^\varepsilon_\flat$ are still approximate solutions on the interval $[0,\tilde{T}]$. Indeed, since t is replaced by $\varepsilon^{-(1/l)}\,t$, the size of the error terms $\dot{\mathbf{f}}^\varepsilon_\flat$ depends on the increase of $\mathbf{f}^\varepsilon_\flat$ with respect to t. At this level, we are faced with secular growth problems [21].

b) Use a convergence process[3] which needs the introduction of an *infinite* cascade of phase shifts. The intuition[4] is that the graph of $\tilde{E}$ becomes continuous (no more gap). This corresponds to the impression of an *infinite* cascade of energy. This remark is consistent with engineering experiments and the observations reported in the statistical approach.

The turbulent phenomena which we study are very complex in their realization. When $t>0$, the description of $\dot{\mathbf{u}}^\varepsilon_\flat(t,\cdot)$ involves an infinite set of phases so that computations and representations are hard to implement. It gives the impression of a chaos. Nevertheless, our analysis reveals that these phenomena contain no mystery in their generation. On the contrary quantitative and qualitative features can be predicted in the framework of non linear geometric optics.

[3]When performing the formal analysis, arbitrary values can be given to the parameters $\varepsilon\in\,]0,1]$ and $l\in\mathbb{N}_*$. For instance ε can be fixed whereas l goes to ∞. Or $l=-(\ln\varepsilon)/(\ln 2)$ so that $\varepsilon^{\frac{1}{l}}\,T=\frac{1}{2}\,T>0$.

[4]Even at a formal level, difficulties occur in order to justify the different convergences. Rigorous results in this direction seem to be a difficult task.

4 Euler equations in the variables (t,x,θ).

As explained in the previous chapter, the demonstration of the Theorem 2.1 is achieved with the representation

$$(4.1) \qquad \tilde{\mathbf{u}}_\flat^\varepsilon(t,x) = \tilde{u}_\flat^\varepsilon\big(t,x,\varepsilon^{-1}\,\varphi_\flat^\varepsilon(t,x)\big)\,, \qquad \tilde{\mathbf{p}}_\flat^\varepsilon(t,x) = \tilde{p}_\flat^\varepsilon\big(t,x,\varepsilon^{-1}\,\varphi_\flat^\varepsilon(t,x)\big)\,.$$

Recall that the complete phase $\varphi_\flat^\varepsilon(t,x)$ is

$$(4.2) \qquad \varphi_\flat^\varepsilon(t,x) \;=\; \varphi_g^\varepsilon(t,x) + \varepsilon\;\varphi_a^\varepsilon(t,x) \;=\; \varphi_0(t,x) + \textstyle\sum_{k=1}^N \varepsilon^{\frac{k}{l}}\;\varphi_k(t,x)$$

and that the profiles $\tilde{u}_\flat^\varepsilon(t,x,\theta)$ and $\tilde{p}_\flat^\varepsilon(t,x,\theta)$ have the form

$$(4.3) \qquad \begin{aligned} \tilde{u}_\flat^\varepsilon(t,x,\theta) \;&=\; \mathbf{u}_0(t,x) + \textstyle\sum_{k=1}^N \varepsilon^{\frac{k}{l}}\;\tilde{U}_k(t,x,\theta)\,, \\ \tilde{p}_\flat^\varepsilon(t,x,\theta) \;&=\; \mathbf{p}_0(t,x) + \textstyle\sum_{k=1}^N \varepsilon^{\frac{k}{l}}\;\tilde{P}_k(t,x,\theta)\,. \end{aligned}$$

4.1 Preliminaries.

• **Anisotropic viscosity.** Mark the abbreviated notations

$$X_\flat^\varepsilon(t,x) := \nabla\varphi_\flat^\varepsilon(t,x) = \textstyle\sum_{k=0}^N \varepsilon^{\frac{k}{l}}\;X_k(t,x)\,, \qquad X_k(t,x) := \nabla\varphi_k(t,x)\,,$$
$$\mathfrak{X}_{\flat 1}^\varepsilon(t,x) := |X_\flat^\varepsilon(t,x)|^{-1}\;X_\flat^\varepsilon(t,x)\,.$$

Complete the unit vector $\mathfrak{X}_{\flat 1}^\varepsilon(t,x)$ into some orthonormal basis of $\mathbb{R}^d$

$$\mathfrak{X}_{\flat i}^\varepsilon(t,x)\cdot\mathfrak{X}_{\flat j}^\varepsilon(t,x) = \delta_{ij}\,, \qquad \forall\,(i,j)\in\{1,\cdots,d\}^2\,,$$

so that all the vector fields $\mathfrak{X}_{\flat i}^\varepsilon$ are smooth functions on $[0,T]\times\mathbb{R}^d$. The corresponding differential operators are denoted

$$\mathfrak{X}_{\flat i}^\varepsilon(\partial) := \mathfrak{X}_{\flat i}^\varepsilon(t,x)\cdot\nabla\,, \qquad i\in\{1,\cdots,d\}\,.$$

Their adjoints are

$$\mathfrak{X}_{\flat i}^\varepsilon(\partial)^* := \mathfrak{X}_{\flat i}^\varepsilon(t,x)\cdot\nabla + \operatorname{div}(\mathfrak{X}_{\flat i}^\varepsilon)(t,x)\,, \qquad i\in\{1,\cdots,d\}\,.$$

Select $\mathfrak{q}\in C_b^\infty([0,T]\times\mathbb{R}^d;S_+^d)$ be such that

$$\exists\,c>0\,; \qquad \mathfrak{q}(t,x)\geq c\,, \qquad \forall\,(t,x)\in[0,T]\times\mathbb{R}^d\,.$$

Let $(m,n)\in\mathbb{N}^2$. Consider the elliptic operator $E_{\flat n}^{\varepsilon m}(\partial)$ defined according to

$$E_{\flat n}^{\varepsilon m}(\partial) \;:=\; \big(\,\varepsilon^{\frac{m}{l}}\,\mathfrak{X}_{\flat 1}^\varepsilon(\partial)^*\,,\,\varepsilon^{\frac{n}{l}}\,\mathfrak{X}_{\flat 2}^\varepsilon(\partial)^*\,,\,\cdots\,,\,\varepsilon^{\frac{n}{l}}\,\mathfrak{X}_{\flat d}^\varepsilon(\partial)^*\,\big)$$
$$\begin{pmatrix} \mathfrak{q}_{11}(t,x) & \mathfrak{q}_{12}(t,x) & \cdots & \mathfrak{q}_{1d}(t,x) \\ \mathfrak{q}_{21}(t,x) & \mathfrak{q}_{22}(t,x) & \cdots & \mathfrak{q}_{2d}(t,x) \\ \vdots & \vdots & & \vdots \\ \mathfrak{q}_{d1}(t,x) & \mathfrak{q}_{d2}(t,x) & \cdots & \mathfrak{q}_{dd}(t,x) \end{pmatrix} \begin{pmatrix} \varepsilon^{\frac{m}{l}}\,\mathfrak{X}_{\flat 1}^\varepsilon(\partial) \\ \varepsilon^{\frac{n}{l}}\,\mathfrak{X}_{\flat 2}^\varepsilon(\partial) \\ \vdots \\ \varepsilon^{\frac{n}{l}}\,\mathfrak{X}_{\flat d}^\varepsilon(\partial) \end{pmatrix}.$$

The introduction of the operator $E^{\varepsilon m}_{\flat n}(\partial)$ in the right of $(\mathcal{E})$ is compatible with the propagation of oscillations only if $m \geq l$ and $n \geq 0$. We retain the limit case $l = m$ and $n = 0$. The other situations are easier to deal with, at least when performing formal computations.

- **Interpretation in** (t,x,θ)**.** To deal with the variables (t,x,θ), define

$$\begin{aligned}
&\mathfrak{d}_{j,\varepsilon} \,:=\, \varepsilon\, \partial_j \,+\, \partial_j \varphi^\varepsilon_\flat \times \partial_\theta \,, \qquad j \in \{0,\cdots,d\}\,, \\
&\mathfrak{d}_\varepsilon \,:=\, (\mathfrak{d}_{1,\varepsilon},\cdots,\mathfrak{d}_{d,\varepsilon})\,, \\
&\mathfrak{grad}^\varepsilon_\flat \,:=\, {}^t(\mathfrak{d}_{1,\varepsilon},\cdots,\mathfrak{d}_{d,\varepsilon}) \,=\, \varepsilon\, \nabla + X^\varepsilon_\flat \times \partial_\theta\,, \\
&\mathfrak{div}^\varepsilon_\flat \,:=\, (\mathfrak{grad}^\varepsilon_\flat)^\star \,=\, \varepsilon\, \mathrm{div} + X^\varepsilon_\flat \cdot \partial_\theta\,.
\end{aligned}$$

The derivatives $\mathfrak{X}^\varepsilon_{\flat\dagger}$ become

$$\begin{aligned}
&\varepsilon\, \mathfrak{X}^\varepsilon_{\flat 1}(\mathfrak{d}_\varepsilon) \,:=\, \varepsilon\, \mathfrak{X}^\varepsilon_{\flat 1}(\partial) \,+\, |X^\varepsilon_\flat(t,x)| \times \partial_\theta\,, \\
&\varepsilon\, \mathfrak{X}^\varepsilon_{\flat j}(\mathfrak{d}_\varepsilon) \,:=\, \varepsilon\, \mathfrak{X}^\varepsilon_{\flat j}(\partial)\,, \qquad \forall\, j \in \{2,\cdots,d\}\,.
\end{aligned}$$

The action of $E^{\varepsilon l}_{\flat 0}(\partial)$ expressed in the variables (t,x,θ) gives rise to some negative differential operator of the order two, noted $E^{\varepsilon l}_{\flat 0}(\mathfrak{d}_\varepsilon)$. The coefficients of the derivatives in $E^{\varepsilon l}_{\flat 0}(\mathfrak{d}_\varepsilon)$ are of size one, except in front of $\mathfrak{X}^\varepsilon_{\flat 1}(\partial)$. To avoid technicalities and to simplify the notations, we substitute the Laplacian $\nu\,\Delta$ for $E^{\varepsilon l}_{\flat 0}(\mathfrak{d}_\varepsilon)$.

When $\nu = 0$, we recover Euler equations. When $\nu > 0$, the action $\nu\,\Delta$ can be viewed as the 'trace' in (t,x,θ) of the anisotropic viscosity $E^{\varepsilon l}_{\flat 0}(\partial)$. Now, consider the Cauchy problem

$$(4.4) \qquad \left\{ \begin{aligned}
&\mathfrak{d}_{0,\varepsilon}\, \tilde u^\varepsilon_\flat + (\tilde u^\varepsilon_\flat \cdot \mathfrak{grad}^\varepsilon_\flat)\, \tilde u^\varepsilon_\flat \;+ \mathfrak{grad}^\varepsilon_\flat\, \tilde p^\varepsilon_\flat \\
&\qquad\qquad\qquad\qquad = \nu\; \varepsilon\; \Delta\, \tilde u^\varepsilon_\flat + \tilde f^\varepsilon_\flat\,, \qquad \mathfrak{div}^\varepsilon_\flat\, \tilde u^\varepsilon_\flat = \tilde g^\varepsilon_\flat\,, \\
&\tilde u^\varepsilon_\flat(0,\cdot) \,=\, \tilde h^\varepsilon_\flat(\cdot)\,,
\end{aligned}\right.$$

with given data

$$\tilde f^\varepsilon_\flat \in \mathcal{W}^\infty_T\,, \qquad \tilde g^\varepsilon_\flat \in \mathcal{W}^\infty_T\,, \qquad \tilde h^\varepsilon_\flat \in H^\infty\,.$$

Suppose that $\nu = 0$ and select some smooth solution $(\tilde u^\varepsilon_\flat, \tilde p^\varepsilon_\flat)$ of (4.4). The expressions $\tilde{\mathbf{u}}^\varepsilon_\flat$ and $\tilde{\mathbf{p}}^\varepsilon_\flat$ given by the formula (4.1) are subjected to

$$(4.5) \qquad \left\{ \begin{aligned}
&\partial_t \tilde{\mathbf{u}}^\varepsilon_\flat + (\tilde{\mathbf{u}}^\varepsilon_\flat \cdot \nabla)\tilde{\mathbf{u}}^\varepsilon_\flat + \nabla \tilde{\mathbf{p}}^\varepsilon_\flat \,=\, \tilde{\mathbf{f}}^\varepsilon_\flat\,, \qquad \mathrm{div}\, \tilde{\mathbf{u}}^\varepsilon_\flat = \tilde{\mathbf{g}}^\varepsilon_\flat\,, \\
&\tilde{\mathbf{u}}^\varepsilon_\flat(0,\cdot) \,=\, \tilde{\mathbf{h}}^\varepsilon_\flat(\cdot)\,,
\end{aligned}\right.$$

where the functions $\tilde{\mathbf{f}}^\varepsilon_\flat(t,x)$, $\tilde{\mathbf{g}}^\varepsilon_\flat(t,x)$ and $\tilde{\mathbf{h}}^\varepsilon_\flat(t,x)$ are obtained by replacing the variable θ by $\varphi^\varepsilon_\flat(t,x)$ in the expressions $\varepsilon^{-1}\, \tilde f^\varepsilon_\flat(t,x,\theta)$, $\varepsilon^{-1}\, \tilde g^\varepsilon_\flat(t,x,\theta)$ and $\tilde h^\varepsilon_\flat(t,x,\theta)$. In other words, any solution of (4.4) with $\nu = 0$ yields a solution of (4.5). From now on, we proceed directly with the relaxed system (4.4).

4.2 The BKW analysis.

Select a smooth solution $\mathbf{u}_0(t,x) \in \mathcal{W}_T^\infty$ of

$$\partial_t \mathbf{u}_0 + (\mathbf{u}_0 \cdot \nabla)\, \mathbf{u}_0 + \nabla \mathbf{p}_0 \;=\; \nu\, \Delta_x \mathbf{u}_0\,, \qquad \text{div}\; \mathbf{u}_0 = 0\,.$$

Choose a phase $\varphi_0(t,x) \in C^1([0,T]\times\mathbb{R}^d)$ with $\nabla\varphi_0(t,x) \in C_b^\infty([0,T]\times\mathbb{R}^d)$. Suppose moreover that it satisfies the eiconal equation (ei) and the condition (2.1). The main step in the construction of approximate solutions is the following intermediate result.

Proposition 4.1. *Select any* $\flat = (l,N) \in \mathbb{N}^2$ *such that* $0 < l < N$. *Consider the following initial data*

$$\begin{aligned}
&\tilde U^*_{k0}(x,\theta) = \Pi_0(0,x)\, \tilde U^*_{k0}(x,\theta) \in H^\infty\,, \qquad 1 \le k \le N\,,\\
&\langle \tilde U_{k0}\rangle(x) \in H^\infty\,, \qquad 1 \le k \le N\,,\\
&\varphi_{k0}(x) \in H^\infty\,, \qquad 1 \le k \le N\,.
\end{aligned}$$

There are finite sequences $\{\tilde U_k\}_{1\le k\le N}$ *and* $\{\tilde P_k\}_{1\le k\le N}$ *with*

$$\tilde U_k(t,x,\theta) \in \mathcal{W}_T^\infty\,, \qquad \tilde P_k(t,x,\theta) \in \mathcal{W}_T^\infty\,, \qquad 1 \le k \le N\,,$$

and a finite sequence $\{\varphi_k\}_{1\le k\le N}$ *with*

$$\varphi_k(t,x) \in \mathcal{W}_T^\infty\,, \qquad 1 \le k \le N\,,$$

which are such that

$$\begin{aligned}
&\Pi_0(0,x)\, \tilde U^*_k(0,x,\theta) = \Pi_0(0,x)\, \tilde U^*_{k0}(x,\theta)\,, \qquad 1 \le k \le N\,,\\
&\langle \tilde U_k\rangle(0,x) = \langle \tilde U_{k0}\rangle(x)\,, \qquad 1 \le k \le N\,,\\
&\varphi_k(0,x) = \varphi_{k0}(x)\,, \qquad 1 \le k \le N\,.
\end{aligned}$$

Define $\varphi^\varepsilon_\flat$ *as in (4.2). All the preceding expressions are adjusted so that the functions* $\tilde u^\varepsilon_\flat$ *and* $\tilde p^\varepsilon_\flat$ *associated with the expansions in (4.3) are approximate solutions on the interval* $[0,T]$. *More precisely, they satisfy (4.4) with*

$$\tilde h^\varepsilon_\flat(x,\theta) \;=\; \mathbf{u}_0(0,x) + \textstyle\sum_{k=1}^N \varepsilon^{\frac{k}{l}}\; \tilde U_{k0}(x,\theta) \tag{4.6}$$

and we have

$$\{\tilde f^\varepsilon_\flat\}_\varepsilon = \bigcirc(\varepsilon^{\frac{N+1}{l}})\,, \qquad \{\tilde g^\varepsilon_\flat\}_\varepsilon = \bigcirc(\varepsilon^{\frac{N+1}{l}})\,. \tag{4.7}$$

• **Proof of the Proposition 4.1.** For convenience, we will drop in this paragraph the tilde '$\tilde{}$' on the profiles $u^\varepsilon_\flat$, $p^\varepsilon_\flat$, U_k and P_k. This modification concerns only this demonstration. We hope that it will not induce confusions: we still work here with the complete phase $\varphi^\varepsilon_\flat$.

Because of (3.1) we can define the application $\Pi^\varepsilon_\flat(t,x)$ which is the orthogonal projector on the hyperplane $\nabla\varphi^\varepsilon_\flat(t,x)^\perp \subset \mathbb{R}^d$. We adopt the convention

$$\Pi^\varepsilon_\flat(t,x) = \textstyle\sum_{k=0}^\infty \varepsilon^{\frac{k}{l}}\ \Pi_k(t,x)\,, \qquad \Pi_k \in \mathcal{W}^\infty_T\,, \qquad \varepsilon \in]0,\varepsilon_0]\,.$$

The access to Π_k needs only the knowledge of the X_j for $j \le k$. Introduce

$$v^\varepsilon_\flat := X^\varepsilon_\flat \cdot u^\varepsilon_\flat = \textstyle\sum_{k=0}^\infty \varepsilon^{\frac{k}{l}}\ V_k\,, \qquad V_k = X_k \cdot \mathbf{u}_0 + \sum_{j=0}^{k-1} X_j \cdot U_{k-j}\,,$$

$$w^\varepsilon_\flat := \Pi^\varepsilon_\flat\, u^\varepsilon_\flat = \textstyle\sum_{k=0}^\infty \varepsilon^{\frac{k}{l}}\ W_k\,, \qquad W_k = \Pi_k\, \mathbf{u}_0 + \sum_{j=0}^{k-1} \Pi_j\, U_{k-j}\,.$$

By construction

$$u^\varepsilon_\flat \ = \ v^\varepsilon_\flat\ |X^\varepsilon_\flat|^{-2}\ X^\varepsilon_\flat + w^\varepsilon_\flat\,, \qquad U_k \ = \ V_k\ |X_0|^{-2}\ X_0\ + W_k + h$$

where h depends only on the X_j for $j \le k$ and on the U_j for $j \le k-1$.

The conditions prescribed in the Proposition 4.1 on the initial data U^*_{k0} allow to fix the functions $\nabla\varphi_0(0,x)\cdot U^*_k(0,x,\theta)$ as we want. Since

$$V^*_k \ = \ \nabla\varphi_0 \cdot U^*_k \ + \ \textstyle\sum_{j=1}^{k-1} X_j \cdot U^*_{k-j}\,,$$

the same is true (by induction) for the components $V^*_k(0,x,\theta)$. To begin with, we impose the polarization conditions

$$\text{(4.8)} \qquad P^*_k \ \equiv \ V^*_k \ \equiv \ 0\,, \qquad \forall\, k \in \{1,\cdots,l\}$$

and we adjust *a priori* the geometrical phase φ^ε_g so that

$$\text{(4.9)} \qquad \partial_t\varphi_k + \bar V_k = 0\,, \qquad \forall\, k \in \{1,\cdots,l-1\}$$

which implies that

$$\partial_t\varphi^\varepsilon_g + (\bar u^\varepsilon_\flat\cdot\nabla)\varphi^\varepsilon_g \ = \ \textstyle\sum_{k=l}^\infty \varepsilon^{\frac{k}{l}}\ \bar V_k \ = \ \bigcirc(\varepsilon)\,.$$

It amounts to the same thing to look at the equations in (4.4) or at the following singular system (we drop here the indices ε and $\flat$ at the level of $u^\varepsilon_\flat$, $v^\varepsilon_\flat$, $w^\varepsilon_\flat$, $p^\varepsilon_\flat$, $\Pi^\varepsilon_\flat$ and $\varphi^\varepsilon_\flat$)

$$\text{(4.10)} \qquad \left\{ \begin{array}{l} \partial_t u + (u\cdot\nabla)\, u + \nabla p \ \ + \varepsilon^{-1}\ (\partial_t\varphi + v)\ \partial_\theta u \\ \qquad\qquad\qquad\qquad\quad + \varepsilon^{-1}\ \partial_\theta p\ \nabla\varphi \ = \ \nu\ \Delta\, u\,, \\ \mathrm{div}\ u + \varepsilon^{-1}\ \partial_\theta v = 0\,. \end{array} \right.$$

The functions v is subjected to

$$\text{(4.11)} \qquad \begin{array}{l} \partial_t v + (u\cdot\nabla)\, v + X\cdot\nabla p + \varepsilon^{-1}\ (\partial_t\varphi + v)\ \partial_\theta v \\ \quad + \varepsilon^{-1}\ \partial_\theta p\ \ \|\ X\ \|^2 - \big(\partial_t X + (u\cdot\nabla)\, X\big)\cdot u = \nu\ X\cdot\Delta\, u\,. \end{array}$$

The functions w satisfies

$$
(4.12) \qquad \begin{aligned} &\partial_t w + (u \cdot \nabla)\, w + \Pi\, \nabla p + \varepsilon^{-1}\, (\partial_t \varphi + v)\, \partial_\theta w \\ &\quad - \big(\partial_t \Pi + (u \cdot \nabla)\, \Pi\big)\, u = \nu\, \Pi\, \Delta\, u\,. \end{aligned}
$$

Substitute the expressions $u^\varepsilon_\flat$ and $p^\varepsilon_\flat$ given by (4.3) into (4.10). Then arrange the terms according to the different powers of ε which are in factor. The contributions coming from the orders $\varepsilon^{\frac{1}{l}-1}, \cdots, \varepsilon^{-\frac{1}{l}}$ and ε^0 are eliminated through (4.8), (4.9) and the constraints imposed on $(\mathbf{u}_0, \mathbf{p}_0)$.

Now, look at the terms in front of $\varepsilon^{\frac{j}{l}}$ with $j \in \mathbb{N}_*$. It remains

$$
(4.13) \qquad \left\{ \begin{aligned} &\partial_t U_j + \textstyle\sum_{k=0}^{j} (U_k \cdot \nabla)\, U_{j-k} + \nabla P_j + \sum_{k=0}^{j} \partial_\theta P_{l+k}\, \nabla \varphi_{j-k} \\ &\qquad + \textstyle\sum_{k=0}^{j-1} (\partial_t \varphi_{l+k} + V_{l+k})\, \partial_\theta U_{j-k} \; = \; \nu\, \Delta\, U_j\,, \\ &\operatorname{div}\, U_j + \partial_\theta V_{j+l} = 0\,. \end{aligned} \right.
$$

Proceed in a similar manner with (4.11). Just arrange the terms which come from $X \cdot \Delta\, u$ and which do not involve U_j in a source term $\mathcal{H}^j_V$

$$
(4.14) \qquad \begin{aligned} &\partial_t V_j + \textstyle\sum_{k=0}^{j} (U_k \cdot \nabla)\, V_{j-k} + \sum_{k=0}^{j-1} (\partial_t \varphi_{l+k} + V_{l+k})\, \partial_\theta V_{j-k} \\ &\quad + \textstyle\sum_{k=0}^{j} X_k \cdot \nabla P_{j-k} \; - \; \sum_{k=0}^{j} \partial_t X_k \cdot U_{j-k} \\ &\quad - \textstyle\sum_{k=0}^{j} \big(\sum_{l=0}^{k} (U_{k-l} \cdot \nabla)\, X_l\big) \cdot U_{j-k} \\ &\quad + \textstyle\sum_{k=1}^{j-1} \big(\sum_{l=0}^{k} X_l \cdot X_{k-l}\big)\, \partial_\theta P_{j+l-k} + |X_0|^2\, \partial_\theta P_{j+l} \\ &\quad = \nu\, X_0 \cdot \Delta\, U_j + \mathcal{H}^j_V(t, x, \theta, U_1, X_1, \cdots, U_{j-1}, X_{j-1}, X_j)\,. \end{aligned}
$$

The same operation with (4.12) yields

$$
(4.15) \qquad \begin{aligned} &\partial_t W_j + \textstyle\sum_{k=0}^{j} (U_k \cdot \nabla)\, W_{j-k} + \sum_{k=0}^{j-1} (\partial_t \varphi_{l+k} + V_{l+k})\, \partial_\theta W_{j-k} \\ &\quad + \textstyle\sum_{k=0}^{j} \Pi_k\, \nabla P_{j-k} \; - \; \sum_{k=0}^{j} \partial_t \Pi_k\, U_{j-k} \\ &\quad - \textstyle\sum_{k=0}^{j} \big(\sum_{l=0}^{k} (U_{k-l} \cdot \nabla)\, \Pi_l\big)\, U_{j-k} \\ &\quad = \nu\, \Pi_0\, \Delta\, U_j + \mathcal{H}^j_W(t, x, \theta, U_1, X_1, \cdots, U_{j-1}, X_{j-1}, X_j)\,. \end{aligned}
$$

Then extract the mean value of (4.13)

$$
(4.16) \qquad \left\{ \begin{aligned} &\partial_t \bar U_j + (\mathbf{u}_0 \cdot \nabla)\, \bar U_j + (\bar U_j \cdot \nabla)\, \mathbf{u}_0 + \nabla \bar P_j \\ &\qquad + \textstyle\sum_{k=1}^{j-1} \langle (U_k \cdot \nabla)\, U_{j-k} \rangle \\ &\qquad + \textstyle\sum_{k=1}^{j-1} \langle V^*_{l+k}\, \partial_\theta U_{j-k} \rangle \; = \; \nu\, \Delta_x\, \bar U_j\,, \\ &\operatorname{div}\, \bar U_j = 0\,. \end{aligned} \right.
$$

Observe also that

$$
(4.17) \qquad V^*_{j+l} \; = \; - \operatorname{div}\, \partial_\theta^{-1} U^*_j\,, \qquad \forall\, j \in \mathbb{N}_*\,.
$$

Consider the inductive reasoning based on

Hypothesis (H_j) :

i) The expressions $U_1, \cdots, U_j$ *and* $P_1, \cdots, P_j$ *are known.*

ii) The phases $\varphi_1, \cdots, \varphi_j$ *are identified. The same is true for the vectors* $X_1, \cdots, X_j$ *and the projectors* $\Pi_1, \cdots, \Pi_j$*. Moreover, the following relations are satisfied*

$$\partial_t \varphi_{j+k} + \bar{V}_{j+k} = 0 \,, \qquad \forall\, k \in \{1, \cdots, l-1\} \,. \tag{4.18}$$

iii) The correctors $V^*_{j+1}, \cdots, V^*_{j+l}$ *and* $P^*_{j+1}, \cdots, P^*_{j+l}$ *are identified and*

$$V^*_{j+k} = -\,\mathrm{div}\ \partial_\theta^{-1} U^*_{j+k-l} \,, \qquad \forall\, k \in \{1, \cdots, l\} \,. \tag{4.19}$$

- <u>Verification</u> <u>of</u> (H_1). The mean value $\bar{U}_1$ is obtained by solving

$$\partial_t \bar{U}_1 + (\mathbf{u}_0 \cdot \nabla)\, \bar{U}_1 + (\bar{U}_1 \cdot \nabla)\, \mathbf{u}_0 + \nabla \bar{P}_1 = \nu\ \Delta_x\, \bar{U}_1 \,, \qquad \mathrm{div}\ \bar{U}_1 = 0 \,.$$

Using (4.9), it allows to determine φ_1. Now look at the oscillating part of (4.15) with the indice $j = 1$. The constraint on $W^*_1 \equiv U^*_1$ writes

$$\partial_t W^*_1 + (\mathbf{u}_0 \cdot \nabla)\, W^*_1 + (\partial_t \varphi_l + \bar{V}_l)\ \partial_\theta W^*_1 = M\, W^*_1 + \nu\ \Pi_0\ \Delta\, W^*_1$$

where M is the linear application

$$M\, U \ := \ (\partial_t \Pi_0)\, U + \big((\mathbf{u}_0 \cdot \nabla) \Pi_0\big)\, U - \Pi_0\, (U \cdot \nabla) \mathbf{u}_0 \,.$$

We impose (4.18) for $j = 1$. In view of (4.9), it reduces to

$$\partial_t \varphi_l + \bar{V}_l = 0 \,.$$

The link between W^*_1 and $\bar{V}_l$ is removed. It remains the linear equation

$$\partial_t W^*_1 + (\mathbf{u}_0 \cdot \nabla)\, W^*_1 = M\, W^*_1 + \nu\ \Pi_0\ \Delta\, W^*_1 \,.$$

When $\nu = 0$, the profile W^*_1 is obtained by integrating along the characteristic curves $\Gamma(\cdot, x)$. This justifies the remark 2.2.4 for $k = 1$. Observe also that the polarization condition $W^*_1 = \Pi_0\, W^*_1$ is conserved since the equation given for W^*_1 is equivalent to

$$\begin{cases} \Pi_0 \left[\partial_t W^*_1 + (\mathbf{u}_0 \cdot \nabla)\, W^*_1 + (W^*_1 \cdot \nabla) \mathbf{u}_0 \right] \ = \ \nu\ \Pi_0\ \Delta\, W^*_1 \,, \\ W^*_1 \ = \ \Pi_0\, W^*_1 \,. \end{cases} \tag{4.20}$$

Introduce the linear form

$$\ell\, U \ := \ |X_0|^{-2} \ \big[\, \partial_t X_0 \cdot U + \big((\mathbf{u}_0 \cdot \nabla) X_0\big) \cdot U - X_0 \cdot \big((U \cdot \nabla) \mathbf{u}_0\big) \,\big] \,.$$

The constraint $V_1^* \equiv 0$ is equivalent to

$$P_{l+1}^* \,=\, \ell\, \partial_\theta^{-1} W_1^* \,+\, \nu\, |X_0|^{-2}\, X_0 \cdot \partial_\theta^{-1}\, \Delta\, W_1^*\,,$$

We have also

$$V_{l+1}^* = \,-\,\mathrm{div}\ \partial_\theta^{-1} W_1^*\,.$$

At this stage, we know who is $U_1 \equiv \bar U_1 + W_1^*$ and $P_1 \equiv 0$. Moreover, we have the relations (4.18) and (4.19). Thus, the hypothesis (H_1) is verified.

- <u>The</u> <u>induction</u>. Suppose that the conditions given in (H_j) are satisfied. The question is to obtain (H_{j+1}). Consider first (4.16) with the indice $j+1$. The relation (4.19) induces simplifications. It remains

$$\left\{ \begin{array}{l} \partial_t \bar U_{j+1} + (\mathbf{u}_0 \cdot \nabla)\, \bar U_{j+1} + (\bar U_{j+1} \cdot \nabla)\, \mathbf{u}_0 \\ \qquad + \sum_{k=1}^{j} (\bar U_k \cdot \nabla)\, \bar U_{j+1-k} + \sum_{k=1}^{j} \mathrm{div}\ \langle U_k^* \otimes U_{j+1-k}^* \rangle \\ \qquad + \nabla \bar P_{j+1} \,=\, \nu\, \Delta_x\, \bar U_{j+1}\,, \qquad \mathrm{div}\ \bar U_{j+1} = 0\,. \end{array} \right. \tag{4.21}$$

This system gives access to $\bar U_{j+1}$ and $\bar P_{j+1}$. For $j=1$, it yields

$$\left\{ \begin{array}{l} \partial_t \bar U_2 + (\mathbf{u}_0 \cdot \nabla)\, \bar U_2 + (\bar U_2 \cdot \nabla)\, \mathbf{u}_0 + \nabla \bar P_2 \\ \qquad + (\bar U_1 \cdot \nabla)\, \bar U_1 + \mathrm{div}\ \langle U_1^* \otimes U_1^* \rangle \,=\, \nu\, \Delta \bar U_2\,, \qquad \mathrm{div}\ \bar U_2 = 0\,. \end{array} \right. \tag{4.22}$$

Because of (2.4), the source term $\langle U_1^* \otimes U_1^* \rangle$ is sure to be non trivial. We recover here that in general $\bar U_2 \not\equiv 0$ even if $\bar U_1(0,\cdot) \equiv \bar U_2(0,\cdot) \equiv 0$. The term $\bar U_2$ excites φ_2 through (3.10). Generically, we have $\varphi_2 \not\equiv 0$ even if

$$\bar U_1(0,\cdot) \,\equiv\, \bar U_2(0,\cdot) \,\equiv\, 0\,, \qquad \varphi_1(0,\cdot) \,\equiv\, \varphi_2(0,\cdot) \,\equiv\, 0\,.$$

Observe however that exceptions can happen (see the remark 3.5.4). The information (4.18) for $k=1$ means that

$$\partial_t \varphi_{j+1} + (\mathbf{u}_0 \cdot \nabla)\, \varphi_{j+1} + X_0 \cdot \bar U_{j+1} + \textstyle\sum_{l=1}^{j}\, X_l \cdot \bar U_{j+1-l} \,=\, 0\,.$$

Deduce φ_{j+1} from this equation, and therefore X_{j+1} and Π_{j+1}. Complete with the triangulation condition

$$\partial_t \varphi_{j+l} + \bar V_{j+l} = 0\,. \tag{4.23}$$

Then extract the oscillating part of (4.15) written with $j+1$. Use (H_j) and (4.23) in order to simplify the resulting equation. It yields

$$\partial_t W_{j+1}^* + (\mathbf{u}_0 \cdot \nabla)\, W_{j+1}^* = M\, W_{j+1}^* + \nu\, \Pi_0\, \Delta\, W_{j+1}^* + f \tag{4.24}$$

where f is known. We get W_{j+1}^* by solving (4.24). Therefore we have U_{j+1}^* and we can deduce $V_{j+l+1}^* = \,-\,\mathrm{div}\ \partial_\theta^{-1} U_{j+1}^*$.

Now look at the constraint (4.14) for the indice $j+1$. Extract the oscillating part. It allows to recover P^*_{j+l+1}. Thus we have (H_{j+1}).

Apply the induction up to $j = N - l$. It yields $U_1, \cdots, U_N$. Construct oscillations $\tilde{u}^\varepsilon_\flat$ and $\tilde{p}^\varepsilon_\flat$ by way of (4.3). It furnishes source terms $\tilde{f}^\varepsilon_\flat$ and $\tilde{g}^\varepsilon_\flat$ through (4.4). By construction, we recover (4.7). □

4.3 Divergence free approximate solutions in (t, x, θ).

In this subsection 4.3, we impose on φ_0 a constraint which is more restrictive than (2.1). We suppose that we can find a direction $\zeta \in \mathbb{R}^d \setminus \{0\}$ such that

$$\exists\, c > 0\,; \qquad \nabla \varphi_0(t,x) \cdot \zeta \,\geq\, c\,, \qquad \forall\, (t,x) \in [0,T] \times \mathbb{R}^d\,. \tag{4.25}$$

Proposition 4.2. *The assumptions are as in the Proposition 4.1. The profiles $\tilde{U}_k$, $\tilde{P}_k$, and the phases φ_k are defined in the same way. Then, there are correctors*

$$c\tilde{u}^\varepsilon_\flat(t,x,\theta) \in \mathcal{W}^\infty_T\,, \qquad \{c\tilde{u}^\varepsilon_\flat\}_\varepsilon \,=\, \bigcirc(\varepsilon^{\frac{N}{l}})$$

such that the functions $\tilde{u}^\varepsilon_\flat$ and $\tilde{p}^\varepsilon_\flat$ defined according to

$$\begin{aligned}
\tilde{u}^\varepsilon_\flat(t,x) &:= \mathbf{u}_0(t,x) + \textstyle\sum_{k=1}^N \varepsilon^{\frac{k}{l}}\ \tilde{U}_k\big(t,x,\varepsilon^{-1}\,\varphi^\varepsilon_g(t,x)\big) + c\tilde{u}^\varepsilon_\flat(t,x) \\
\tilde{p}^\varepsilon_\flat(t,x) &:= \mathbf{p}_0(t,x) + \textstyle\sum_{k=1}^N \varepsilon^{\frac{k}{l}}\ \tilde{P}_k\big(t,x,\varepsilon^{-1}\,\varphi^\varepsilon_g(t,x)\big)
\end{aligned}$$

satisfy the Cauchy problem

$$\left\{\begin{aligned}
&\mathfrak{d}_{0,\varepsilon}\,\tilde{u}^\varepsilon_\flat + (\tilde{u}^\varepsilon_\flat \cdot \mathfrak{grad}^\varepsilon_\flat)\,\tilde{u}^\varepsilon_\flat + \mathfrak{grad}^\varepsilon_\flat\,\tilde{p}^\varepsilon_\flat \\
&\qquad\qquad = \nu\;\varepsilon\;\Delta\,\tilde{u}^\varepsilon_\flat + \tilde{f}^\varepsilon_\flat\,, \qquad \mathfrak{div}^\varepsilon_\flat\,\tilde{u}^\varepsilon_\flat = 0 \\
&\tilde{u}^\varepsilon_\flat(0,x,\theta) \,=\, \mathbf{u}_0(0,x) + \textstyle\sum_{k=1}^N \varepsilon^{\frac{k}{l}}\ \tilde{U}_{k0}(x,\theta)
\end{aligned}\right. \tag{4.26}$$

and we still have $\{\tilde{f}^\varepsilon_\flat\}_\varepsilon \,=\, \bigcirc(\varepsilon^{\frac{N+1}{l}})$.

We need some material before proving the Proposition 4.2.

• **The divergence free relation in the variables (t,x,θ).** We can select some special right inverse of the application $\mathfrak{div}^\varepsilon_\flat\,:\, H^{\infty *}_T \longrightarrow H^{\infty *}_T$.

Lemma 4.1. *There is a linear operator* $\mathfrak{ridiv}^\varepsilon_\flat\,:\,\mathrm{Im}\,(\mathfrak{div}^\varepsilon_\flat) \longrightarrow H^{\infty *}_T$ *with*

$$\mathfrak{div}^\varepsilon_\flat \circ \mathfrak{ridiv}^\varepsilon_\flat\ g \,=\, g\,, \qquad \forall\, g \in \mathrm{Im}\,(\mathfrak{div}^\varepsilon_\flat)\,. \tag{4.27}$$

For all $m \in \mathbb{N}$, there is a constant $C_m > 0$ such that

$$\|\ \mathfrak{ridiv}^\varepsilon_\flat\ g\,\|_{H^m} \leq C_m\ \ \|\,g\,\|_{H^{m+1+\frac{d}{2}}}\,, \qquad \forall\, g \in \mathrm{Im}\,(\mathfrak{div}^\varepsilon_\flat)\,. \tag{4.28}$$

Proof of the Lemma 4.1. Let $n \in \mathbb{N}_*$. Note

$$t_j := j\,T/n\,, \qquad x_j = k/n\,, \qquad 1 \leq j \leq n-1\,, \qquad k \in \mathbb{Z}^d\,.$$

Consider a related partition of unity

$$\begin{aligned}
&\chi_{(j,k)} \in C^\infty([0,T] \times \mathbb{R}^d)\,, \qquad\qquad (j,k) \in \{1,\cdots,n-1\} \times \mathbb{Z}^d\,,\\
&\textstyle\sum_{j=1}^{n-1} \sum_{k \in \mathbb{Z}^d} \chi_{(j,k)}(t,x) \,=\, 1\,, \qquad \forall\,(t,x) \in [0,T] \times \mathbb{R}^d\,,\\
&\big\{\,(t,x)\,;\ \chi_{(j,k)}(t,x) \neq 0\,\big\} \,\subset\, [t_j - \tfrac{2}{n}, t_j + \tfrac{2}{n}] \times B(x_j, \tfrac{2}{n}]\,,\\
&\big\{\,(t,x)\,;\ \chi_{(j,k)}(t,x) = 1\,\big\} \,\supset\, [t_j - \tfrac{1}{n}, t_j + \tfrac{1}{n}] \times B(x_j, \tfrac{1}{n}]\,.
\end{aligned}$$

By hypothesis, there is a function $v \in H^{\infty *}_T$ such that $g = \mathfrak{div}^\varepsilon_\flat\, v$. Introduce

$$v_{(j,k)} \,:=\, \chi_{(j,k)}\, v \in H^{\infty *}_T\,, \qquad g_{(j,k)} := \mathfrak{div}^\varepsilon_\flat\, v_{(j,k)}\,.$$

It suffices to exhibit $\mathfrak{ridiv}^\varepsilon_\flat\, g_{(j,k)}$ and to show (4.28) with a constant C_m which is uniform in (j,k). The problem of finding $\mathfrak{ridiv}^\varepsilon_\flat\, g_{(j,k)}$ can be reduced to a model situation. This can be achieved by using a change of variables in (t,x), based on (4.25). From now on, the time t is viewed as a parameter, the space variable is $x = (x_1, \hat{x}) \in \mathbb{R} \times \mathbb{R}^{d-1}$, and we work with

$$\begin{aligned}
&g = g^* = \mathfrak{div}^\varepsilon_\flat\, v = (\varepsilon\,\partial_1 + \partial_\theta) v_1 + \partial_2 v_2 + \cdots + \partial_d v_d\,,\\
&\big\{\,x\,;\ g(x,\theta) \neq 0\,\big\} \,\subset\, \big\{\,x\,;\ v(x,\theta) \neq 0\,\big\} \,\subset\, B(0, \tfrac{1}{2}]\,.
\end{aligned}$$

Let $\psi \in C^\infty(\mathbb{R}^{d-1}; \mathbb{R}^+)$ be such that $\int_{\mathbb{R}^{d-1}} \psi(\hat{x})\ d\hat{x} = 1$ and

$$\big\{\,\hat{x}\,;\ \psi(\hat{x}) \neq 0\,\big\} \,\subset\, B(0,1]\,, \qquad \big\{\,\hat{x}\,;\ \psi(\hat{x}) = 1\,\big\} \,\supset\, B(0, \tfrac{1}{2}]\,.$$

Decompose g according to

$$g \,=\, (g - \breve{g})\ \psi + \breve{g}\ \psi\,, \qquad \breve{g}(x) := \textstyle\int_{\mathbb{R}^{d-1}} g(x_1, \hat{x})\ d\hat{x} \,=\, (\varepsilon\,\partial_1 + \partial_\theta)\breve{v}_1\,.$$

Seek a special solution u having the form

$$u \,=\, \mathfrak{ridiv}^\varepsilon_\flat\, g \,=\, {}^t\big(\,a\,,\, \mathrm{ridiv}\,[(g - \breve{g})\ \psi]\,\big)\,, \qquad a \in H^{\infty *}_T$$

where 'ridiv' is the operator of Lemma 2.1 applied in the dimension $d-1$. It remains to control the scalar function a which satisfies the constraint

$$\varepsilon\ \partial_1 a + \partial_\theta a \,=\, h \,:=\, \breve{g}\ \psi \,=\, (\varepsilon\ \partial_1 + \partial_\theta)(\breve{v}_1\, \psi)\,.$$

Take the explicit solution

$$\begin{aligned}
a(x_1, \hat{x}, \theta) \,&=\, \textstyle\int_{-\infty}^\theta h\big(x_1 + \varepsilon\,(s - \theta), \hat{x}, s\big)\ ds\\
&=\, \varepsilon^{-1}\ \textstyle\int_{-\infty}^0 h\big(x_1 + r, \hat{x}, \theta + \varepsilon^{-1}\, r\big)\ dr\,.
\end{aligned}$$

By construction

$$a(x, \theta + 1) = a(x,\theta)\,, \qquad \textstyle\int_{\mathbb{T}} a(x,\theta)\ d\theta \,=\, 0\,, \qquad \forall\,(x,\theta) \in \mathbb{R}^d \times \mathbb{T}\,.$$

For $|x_1| + |\hat{x}| \geq 2$, we find

$$a(x,\theta) \,=\, \int_{-\infty}^{\theta} \frac{d}{ds} \big[(\check{v}_1 \, \psi)\big(x_1 + \varepsilon\,(s-\theta), \hat{x}, s\big)\big] \; ds \,=\, (\check{v}_1 \, \psi)(x,\theta) \,=\, 0 \, .$$

It implies that

$$\big\{ \, (x,\theta) \, ; \; a(x,\theta) \neq 0 \, \big\} \; \subset \; B(0;2] \, .$$

Note $\mathfrak{h} := \partial_\theta^{-1} h \in H^{\infty *}$. Obviously

$$\| \, \mathfrak{h} \, \|_{H^m} \leq C_m \;\; \| \, h \, \|_{H^m} \, , \qquad \forall \, m \in \mathbb{N} \, ,$$

$$\big\{ \, (x,\theta) \, ; \; \mathfrak{h}(x,\theta) \neq 0 \, \big\} \; \subset \; B(0;1] \, ,$$

and we have the identity

$$a(x_1,\hat{x},\theta) \,=\, \mathfrak{h}(x_1,\hat{x},\theta) \,-\, \int_{-x_1-1}^{-x_1+1} \; \partial_1 \mathfrak{h}\big(x_1 + r, \hat{x}, \theta + \varepsilon^{-1} \, r\big) \; dr \, .$$

The term on the right is supported in $B(0,2]$. Use Fubini and Cauchy-Schwarz inequality to control the integration of $\partial_1 \mathfrak{h}$. It yields (4.28). $\Diamond$

• **The Leray projector interpreted in the variables** (t,x,θ)**.** Introduce the closed subspace

$$\mathrm{F}_\flat^\varepsilon \, := \, \big\{ \, u^* \in L^{2*}_T \, ; \; \mathfrak{div}_\flat^\varepsilon \, u^* = 0 \, \big\} \; \subset \; L^{2*}_T \, .$$

Note $\mathfrak{P}_\flat^\varepsilon$ the orthogonal projector from L^{2*}_T onto $\mathrm{F}_\flat^\varepsilon$. This is a self-adjoint operator such that

$$\ker \, \mathfrak{div}_\flat^\varepsilon = \mathrm{Im} \, \mathfrak{P}_\flat^\varepsilon \, , \qquad \mathrm{Im} \, \mathfrak{grad}_\flat^\varepsilon = (\ker \, \mathfrak{div}_\flat^\varepsilon)^\perp = \ker \, \mathfrak{P}_\flat^\varepsilon \, .$$

Expand the function $u^* \in L^{2*}_T$ in Fourier series and decompose the action of $\mathfrak{P}_\flat^\varepsilon$ in view of the Fourier modes

$$u^*(t,x,\theta) = \textstyle\sum_{k \in \mathbb{Z}_*} \mathbf{u}_k(t,x) \; e^{i\,k\,\theta} \, , \qquad \mathfrak{P}_\flat^\varepsilon \, u^* = \sum_{k \in \mathbb{Z}_*} \mathfrak{P}_{\flat k}^\varepsilon \, \mathbf{u}_k(t,x) \; e^{i\,k\,\theta} \, .$$

Simple computations indicate that

$$\mathfrak{P}_{\flat k}^\varepsilon \, \mathbf{u}_k \, := \, e^{-\,i\,\varepsilon^{-1}\,k\,\varphi_\flat^\varepsilon} \; \Pi(D_x) \; \big(e^{i\,\varepsilon^{-1}\,k\,\varphi_\flat^\varepsilon} \, \mathbf{u}_k \big) \, .$$

The following result explains why the projector $\mathfrak{P}_\flat^\varepsilon$ is replaced by Π_0 when performing the BKW calculus.

Lemma 4.2.

i) The family $\{\mathfrak{P}_\flat^\varepsilon\}_\varepsilon$ *is in* $\mathfrak{UL}^0$. *We have* $[\partial_\theta ; \mathfrak{P}_\flat^\varepsilon] = 0$ *and*

$$[\mathfrak{d}_{j,\varepsilon} ; \mathfrak{P}_\flat^\varepsilon] = 0 \, , \qquad \forall \, j \in \{0, \cdots, d\} \, .$$

ii) The projector $\Pi_\flat^\varepsilon(t,x)$ *is an approximation of* $\mathfrak{P}_\flat^\varepsilon$ *in the sense that*

$$\big\{ \, \mathfrak{P}_\flat^\varepsilon - \Pi_\flat^\varepsilon \, \big\}_\varepsilon \, \in \, \varepsilon \; \mathfrak{UL}^{2+\frac{d}{2}} \, , \qquad \big\{ \, \mathfrak{P}_\flat^\varepsilon \; (\mathrm{Id} - \Pi_\flat^\varepsilon) \, \big\}_\varepsilon \, \in \, \varepsilon \; \mathfrak{UL}^1 \, .$$

Proof of the Lemma 4.2. Since $\mathfrak{P}_\flat^\varepsilon$ is a projector, we are sure that

$$\| \mathfrak{P}_\flat^\varepsilon u \|_{L^2_T} \leq \| u \|_{L^2_T}, \qquad \forall (\varepsilon, u) \in]0, \varepsilon_0] \times L^2_T .$$

It shows that $\{\mathfrak{P}_\flat^\varepsilon\}_\varepsilon \in \mathfrak{UL}^0$. Compute

$$[\mathfrak{d}_{j,\varepsilon}; \mathfrak{P}_\flat^\varepsilon]\, u^*(t,x,\theta) \;=\; \textstyle\sum_{k\in\mathbb{Z}_*} [\, \varepsilon\ \partial_j + i\ k\ \partial_j \varphi_\flat^\varepsilon\, ;\, \mathfrak{P}_{\flat k}^\varepsilon\,]\, \mathbf{u}_k(t,x)\ e^{i\,k\,\theta} .$$

Observe that

$$\begin{aligned}(\varepsilon\ \partial_j + i\ k\ \partial_j \varphi_\flat^\varepsilon)\ \mathfrak{P}_{\flat k}^\varepsilon\, \mathbf{u}_k \;&=\; e^{-\,i\,\varepsilon^{-1}\,k\,\varphi_\flat^\varepsilon}\ \Pi(D_x)\ \varepsilon\,\partial_j\ \big(e^{i\,\varepsilon^{-1}\,k\,\varphi_\flat^\varepsilon}\ \mathbf{u}_k\big) \\ &=\; \mathfrak{P}_{\flat k}^\varepsilon\ (\varepsilon\ \partial_j + i\ k\ \partial_j \varphi_\flat^\varepsilon)\, \mathbf{u}_k .\end{aligned}$$

All these informations give access to the first assertion i). Now consider ii). The asymptotic expansion formula for pseudodifferential operators say that for all $\mathbf{u}_k$ in $C_0^\infty(\mathbb{R}_T^d)$ we have

$$\forall\,(t,x) \in \mathbb{R}_T^d, \qquad \lim_{\varepsilon \longrightarrow 0}\ \big\{\, (\mathfrak{P}_{\flat k}^\varepsilon\, \mathbf{u}_k)(t,x) \;-\; \Pi\big(\nabla \varphi_\flat^\varepsilon(t,x)\big)\, \mathbf{u}_k(t,x)\,\big\} \;=\; 0 .$$

Since $\Pi_\flat^\varepsilon = \Pi(\nabla\varphi_\flat^\varepsilon)$, it indicates that $\mathfrak{P}_\flat^\varepsilon$ is close to $\Pi_\flat^\varepsilon$. We have to make this information more precise. To this end, proceed to the decomposition

$$u^* \;=\; v^* + \varepsilon\ \nabla p^* + \partial_\theta p^* \times X_\flat^\varepsilon , \qquad v^* = \mathfrak{P}_\flat^\varepsilon\, u^* .$$

We seek a solution (v^*, p^*) of these constraints such that

$$v^* \;=\; \Pi_\flat^\varepsilon\, u^* + \varepsilon\ \tilde{v}^* , \qquad p^* \;= \| X_\flat^\varepsilon \|^{-2}\ \ X_\flat^\varepsilon \cdot \partial_\theta^{-1} u^* + \varepsilon\ \tilde{p}^* .$$

After substitution, we find the relation

$$-\,\nabla\big(\| X_\flat^\varepsilon \|^{-2}\ \ X_\flat^\varepsilon \cdot \partial_\theta^{-1} u^*\big) \;=\; \tilde{v}^* + \varepsilon\ \nabla\tilde{p}^* + \partial_\theta \tilde{p}^* \times X_\flat^\varepsilon$$

which must be completed by the condition

$$-\,\mathrm{div}\,(\Pi_\flat^\varepsilon\, u^*) \;=\; \varepsilon\ \mathrm{div}\,\tilde{v}^* + X_\flat^\varepsilon \cdot \partial_\theta \tilde{v}^* .$$

It follows that

$$\tilde{v}^* \;=\; -\,\mathfrak{P}_\flat^\varepsilon\, \big[\nabla(\| X_\flat^\varepsilon \|^{-2}\ \ X_\flat^\varepsilon \cdot \partial_\theta^{-1} u^*)\big] + (\mathfrak{P}_\flat^\varepsilon - \mathrm{Id})\ \mathfrak{ridiv}_\flat^\varepsilon\ \big(\mathrm{div}\,(\Pi_\flat^\varepsilon\, u^*)\big) .$$

In view of this relation, the point ii) becomes clear. $\Diamond$

Consider the Cauchy problem

$$\mathfrak{d}_{0,\varepsilon} u^* + \varepsilon^{-1}\ \mathfrak{grad}_\flat^\varepsilon\, p^* \;=\; f^* , \qquad \mathfrak{div}_\flat^\varepsilon\, u^* = 0 , \qquad u^*(0,\cdot) = h^*(\cdot)$$

with data $f^* \in L_T^{2*}$ and $h^* \in L^{2*}$. Compose on the left with $\mathfrak{P}_\flat^\varepsilon$. It yields

$$\mathfrak{d}_{0,\varepsilon} u^* = \mathfrak{P}_\flat^\varepsilon\, f^* + [\mathfrak{d}_{0,\varepsilon}; \mathfrak{P}_\flat^\varepsilon]\, u^* , \qquad u^*(0,\cdot) = \mathfrak{P}_\flat^\varepsilon\, h^*(\cdot) .$$

The Cauchy problem can be solved in two steps. First extract u^* from the above equation. Then recover p^* from the remaining relations.

• **Proof of the Proposition 4.2.** It remains to absorb the term $\tilde{g}^\varepsilon_\flat$. Use the decomposition

$$g^\varepsilon_\flat = \langle \tilde{g}^\varepsilon_\flat \rangle + \tilde{g}^{\varepsilon *}_\flat \,, \qquad \langle \tilde{g}^\varepsilon_\flat \rangle \in \mathrm{Im}\,(\mathrm{div})\,, \qquad \tilde{g}^{\varepsilon *}_\flat \in \mathrm{Im}\,(\mathfrak{div}^\varepsilon_\flat)\,.$$

It suffices to choose

$$c\mathbf{u}^\varepsilon_\flat \,:=\, -\,\mathrm{ridiv}\,\langle \tilde{g}^\varepsilon_\flat \rangle - \mathfrak{ridiv}^\varepsilon_\flat\, \tilde{g}^{\varepsilon *}_\flat = \bigcirc(\varepsilon^{\frac{N+1}{l}})\,.$$

5 Stability of strong oscillations

The case of turbulent regimes ($l \geq 3$) will not be undertaken here. From now on, fix $l = 2$ and $N \gg (6+d)$. Consider the Cauchy problem

$$(5.1) \qquad \begin{cases} \partial_t \mathbf{u}^\varepsilon + (\mathbf{u}^\varepsilon \cdot \nabla)\,\mathbf{u}^\varepsilon + \nabla \mathbf{p}^\varepsilon \,=\, \nu\, E^{\varepsilon l}_{\flat 0}(\partial)\,\mathbf{u}^\varepsilon\,, \qquad \mathrm{div}\,\mathbf{u}^\varepsilon \,=\, 0\,, \\ \mathbf{u}^\varepsilon(0,x) \,=\, \mathbf{u}^\varepsilon_\flat(0,x)\,. \end{cases}$$

Let T_ε be the upper bound of the $T \geq 0$ such that (5.1) has a solution $\mathbf{u}^\varepsilon \in \mathcal{W}^0_T$. Classical results [4] for fluid equations imply that $T_\varepsilon > 0$. Our aim in this chapter 5 is to investigate the singular limit '*ε goes to zero*'. Such an analysis must at least contain the two following parts.

a) An *existence* result for a time T_0 which is independent on the small parameter $\varepsilon \in]0, \varepsilon_0]$. It is required that

$$\inf\, \{\, T_\varepsilon\,;\; \varepsilon \in]0,1]\,\} \;\geq\; T_0 \;>\; 0\,.$$

When $\nu > 0$, or when $\nu = 0$ and $d = 2$, we know [4]-[23] that $T_\varepsilon = +\infty$ so that $T_0 = +\infty$. When $\nu = 0$ and $d \geq 3$, nothing guarantees that $T_0 > 0$. To our knowledge, this is an open question.

b) A *convergence* result. The exact solution $\mathbf{u}^\varepsilon$ is not sure to remain close on the whole interval $[0, T_0]$ to the approximate solution $\mathbf{u}^\varepsilon_\flat$ given by the Theorem 2.1. Proving estimates on $\mathbf{u}^\varepsilon - \mathbf{u}^\varepsilon_\flat$ is a delicate matter.

5.1 Various types of instabilities

• **Obvious instabilities.** The obvious instabilities are the mechanisms of amplifications which can be detected by looking directly at the formal expansions $\mathbf{u}^\varepsilon_\flat$. They imply the non linear instability of Euler equations. Indeed, fix any $T > 0$, any $\mathbf{u}_0 \in \mathcal{W}^\infty_T(\mathbb{R}^d)$ which is solution of $(\mathcal{E})$, and any $\delta > 0$. Work on the balls

$$B_0(\mathbf{u}_0;\delta] \,:=\, \big\{\, \mathbf{u} \in L^2\,;\; \|\, \mathbf{u}(\cdot) - \mathbf{u}_0(0,\cdot)\,\|_{L^2(\mathbb{R}^d)} \leq \,\delta \,\big\}\,.$$

$$B_T(\mathbf{u}_0;\delta] \,:=\, \big\{\, \mathbf{u} \in L^2_T\,;\; \|\, \mathbf{u} - \mathbf{u}_0\,\|_{L^2([0,T]\times\mathbb{R}^d)} \leq \,\delta \,\big\}\,.$$

Proposition 5.1. *For all constant $C > 0$, there are small data*

$$(\mathbf{h}, \tilde{\mathbf{h}}) \in \big(B_0(\mathbf{u}_0; \delta] \cap H^\infty \big)^2 , \qquad (\mathbf{f}, \tilde{\mathbf{f}}) \in \big(B_T(\mathbf{u}_0; \delta] \cap \mathcal{W}^\infty_T \big)^2$$

so that the Cauchy problems

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} + \nabla \mathbf{p} \, = \, \mathbf{f} \, , \qquad \mathrm{div}\ \mathbf{u} \, = \, 0 \, , \qquad \mathbf{u}(0,\cdot) = \mathbf{h}(\cdot) \, ,$$
$$\partial_t \tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla)\, \tilde{\mathbf{u}} + \nabla \tilde{\mathbf{p}} \, = \, \tilde{\mathbf{f}} \, , \qquad \mathrm{div}\ \tilde{\mathbf{u}} \, = \, 0 \, , \qquad \tilde{\mathbf{u}}(0,\cdot) = \tilde{\mathbf{h}}(\cdot) \, ,$$

have solutions $(\mathbf{u}, \tilde{\mathbf{u}}) \in B_T(\mathbf{u}_0; \delta]^2$ and there is $t \in]0, T]$ such that

$$\begin{aligned} \| \, (\mathbf{u} - \tilde{\mathbf{u}})(t,\cdot) \, \|_{L^2(\mathbb{R}^d)} \, \geq \, C \, \Big(\| \, \mathbf{h} - \tilde{\mathbf{h}} \, \|_{L^2(\mathbb{R}^d)} & \\ + \textstyle\int_0^t \, \| \, (\mathbf{f} - \tilde{\mathbf{f}})(s,\cdot) \, \|_{L^2(\mathbb{R}^d)} \ ds \Big) \, . & \end{aligned} \tag{5.2}$$

Inequalities as (5.2) are well-known. In general [7]-[15]-[17], the demonstration is achieved in two steps. First detect equilibria where instability arises in the discrete spectrum. Then establish that linearized instability implies non linear instability. The procedure we adopt below is different. We just look at approximate solutions like $\mathbf{u}^\varepsilon_\flat$. It follows a more simple proof of (5.2).

Proof of the Proposition 5.1. Take $l = 2$ and $N \geq (8 + d)$. Consider two deals of initial data

$$\tilde{U}^1_k(0, x, \theta) \, , \qquad \varphi^1_k(0, x) \, , \qquad 1 \leq k \leq N \, ,$$
$$\tilde{U}^2_k(0, x, \theta) \, , \qquad \varphi^2_k(0, x) \, , \qquad 1 \leq k \leq N \, .$$

Fix these expressions in the following way

$$\tilde{U}^1_1(0,\cdot) \equiv \tilde{U}^2_1(0,\cdot) \, , \qquad \varphi^1_1(0,\cdot) \equiv \varphi^2_1(0,\cdot) \equiv 0 \, , \qquad \varphi^1_2(0,\cdot) \equiv \varphi^2_2(0,\cdot) \equiv 0 \, .$$

It implies that

$$\tilde{U}^1_1(t,\cdot) \equiv \tilde{U}^2_1(t,\cdot) \, , \qquad \varphi^1_1(t,\cdot) \equiv \varphi^2_1(t,\cdot) \equiv 0 \, , \qquad \forall \, t \in [0, T] \, .$$

Adjust $\tilde{U}^1_2(0,\cdot)$ and $\tilde{U}^2_2(0,\cdot)$ so that

$$\partial_t \, (\varphi^1_2 - \varphi^2_2)(0,\cdot) \, = \, - \nabla \varphi_0 \cdot \langle \tilde{U}^1_2 - \tilde{U}^2_2 \rangle (0,\cdot) \, \not\equiv \, 0 \, .$$

Therefore, we are sure to find some $t > 0$ such that $(\varphi^1_2 - \varphi^2_2)(t,\cdot) \not\equiv 0$. It follows that

$$\begin{aligned} U^1_1(t,x,\theta) \, &= \, \tilde{U}^1_1\big(t,x,\theta + \varphi^1_2(t,x)\big) \\ \not\equiv \, U^2_1(t,x,\theta) \, &= \, \tilde{U}^1_1\big(t,x,\theta + \varphi^2_2(t,x)\big) \, . \end{aligned} \tag{5.3}$$

Note $\mathbf{u}^{\varepsilon 1}_\flat$ and $\mathbf{u}^{\varepsilon 2}_\flat$ the approximate solutions built with the profiles $\{U^1_k\}_k$ and $\{U^2_k\}_k$. The associated error terms are $\mathbf{f}^{\varepsilon 1}_\flat$ and $\mathbf{f}^{\varepsilon 2}_\flat$.

Proceed by contradiction. Suppose that the Proposition 5.1 is wrong. Then, there is $C > 0$ and $\varepsilon_1 \in]0, \varepsilon_0]$ such that for all $\varepsilon \in]0, \varepsilon_1]$, we have

$$\| (\mathbf{u}_\flat^{\varepsilon 1} - \mathbf{u}_\flat^{\varepsilon 2})(t,\cdot) \|_{L^2(\mathbb{R}^d)} \leq C \, \Big(\| (\mathbf{u}_\flat^{\varepsilon 1} - \mathbf{u}_\flat^{\varepsilon 2})(0,\cdot) \|_{L^2(\mathbb{R}^d)} + \int_0^t \| (\mathbf{f}_\flat^{\varepsilon 1} - \mathbf{f}_\flat^{\varepsilon 2})(s,\cdot) \|_{L^2(\mathbb{R}^d)} \, ds \Big) \, .$$

Divide this inequality by $\sqrt{\varepsilon}$. By construction, we have

$$\varepsilon^{-\frac{1}{2}} \ \| (\mathbf{u}_\flat^{\varepsilon 1} - \mathbf{u}_\flat^{\varepsilon 2})(0,\cdot) \|_{L^2(\mathbb{R}^d)} = \bigcirc(\sqrt{\varepsilon}) \, ,$$

$$\varepsilon^{-\frac{1}{2}} \ \| (\mathbf{f}_\flat^{\varepsilon 1} - \mathbf{f}_\flat^{\varepsilon 2})(s,\cdot) \|_{L^2(\mathbb{R}^d)} = \bigcirc(\sqrt{\varepsilon}) \, , \qquad \forall \, s \in [0,t] \, .$$

$$\varepsilon^{-\frac{1}{2}} \ \| (\mathbf{u}_\flat^{\varepsilon 1} - \mathbf{u}_\flat^{\varepsilon 2})(t,\cdot) \|_{L^2(\mathbb{R}^d)} = \| (U_1^1 - U_1^2)(t,\cdot,\varepsilon^{-1} \, \varphi_g^\varepsilon(t,\cdot) \|_{L^2(\mathbb{R}^d)} + \bigcirc \, (\sqrt{\varepsilon}) \, .$$

It follows that

$$\lim_{\varepsilon \longrightarrow 0} \ \varepsilon^{-\frac{1}{2}} \ \| (\mathbf{u}_\flat^{\varepsilon 1} - \mathbf{u}_\flat^{\varepsilon 2})(t,\cdot) \|_{L^2(\mathbb{R}^d)} = \| (U_1^1 - U_1^2)(t,\cdot) \|_{L^2(\mathbb{R}^d \times \mathbb{T})} = 0$$

which is inconsistent with (5.3). □

Remark 5.1.1: In the demonstration presented above, the amplification is due to φ_2 which is the principal term in the adjusting phase. The presence of φ_2 becomes efficient in comparison with the other effects when

$$| \, \tilde{U}_1^1\big(t,x,\theta + \varphi_2^1(t,x)\big) - \tilde{U}_1^2\big(t,x,\theta + \varphi_2^2(t,x)\big) \, | \ \sim \ c \ t \ \gg \ \sqrt{\varepsilon} \, .$$

This requires to wait a lapse of time bigger than $\sqrt{\varepsilon}$. This delay can be reduced by adapting the above procedure to the cases $l > 2$. △

Obvious instabilities have an important consequence. To describe the related amplifications, it is necessary to introduce new quantities which correspond to the phase shifts. In other words, the only way to get L^2 −estimates is to *blow up* the state variables. This principle is detailed in [6] in the case of compressible Euler equations.

• **Hidden instabilities.** Hidden instabilities are the amplifications which are not detected by the monophase description of the section 4. On the other hand, they can be revealed by a multiphase analysis. Introduce a second phase $\psi_0(t,x) \in \mathcal{W}_T^\infty$ such that

$$\partial_t \psi_0 + (\mathbf{u}_0 \cdot \nabla) \, \psi_0 = 0 \, , \qquad \nabla \psi_0 \wedge \nabla \varphi_0 \not\equiv 0$$

and disturb the Cauchy data of (5.1) according to

$$\mathbf{u}^\varepsilon(0,x) \ = \ \mathbf{u}_\flat^\varepsilon(0,x) \ + \ \varepsilon^{\frac{M}{l}} \ U\big(x, \varepsilon^{-1} \ \psi_0(0,x)\big) \, , \qquad M \gg N \, .$$

The small oscillations contained in the perturbation of size $\varepsilon^{\frac{M}{l}}$ are not always kept under control. They interact with $\mathbf{u}_\flat^\varepsilon$ and with themselves. They can be organized in such a way to affect the leading oscillation $\mathbf{u}_\flat^\varepsilon$. Concretely (see [7]), we can adjust U and ψ_0 so that there is a constant $C > 0$ and times $t_\varepsilon \in]0, T_\varepsilon[$ going to zero with ε such that

$$\| (\mathbf{u}^\varepsilon - \mathbf{u}_\flat^\varepsilon)(t, \cdot) \|_{L^2(\mathbb{R}^d)} \geq C \, \varepsilon^{\frac{1}{2}} , \qquad \forall \, \varepsilon \in]0, \varepsilon_0] \, .$$

The power $\varepsilon^{\frac{M}{l}}$ at the time $t = 0$ is turned into $\varepsilon^{\frac{1}{2}}$ at the time $t = t_\varepsilon$. Such amplifications occur whatever the selection of $l \geq 2$. They imply minorations like (5.2). However, the underlying mechanisms are distinct from the preceding ones. They are implemented by oscillations which are transversal to φ_0 and whose wavelengths are $\bigcirc(\varepsilon)$. They are cancelled by the addition of the anisotropic viscosity $\nu \, E_{\flat 0}^{\varepsilon l}$.

5.2 Exact solutions

• **Statement of the result.** The first information brought by the BKW construction is that mean values $\bar{U}_k$ and oscillations U_k^* of the profiles U_k do not play the same part. This fact is well illustrated by the rules of transformation (3.5). It means that we have to distinguish these quantities if we want to go further in the analysis. This can be done by involving the variables (t, x, θ) that is by working at the level of (4.26). To deal with $(u^\varepsilon, p^\varepsilon)(t, x, \theta)$ instead of $(\mathbf{u}^\varepsilon, \mathbf{p}^\varepsilon)(t, x)$ is usual in non linear geometric optics [25]. It allows to mark the terms apt to induce instabilities.

Select some approximate solution $(u^\varepsilon_{(2,N)}, p^\varepsilon_{(2,N)})$ with source term $f^\varepsilon_{(2,N)}$ given by the Proposition 4.2 and look at

$$(5.4) \qquad \left\{ \begin{array}{l} \mathfrak{d}_{0,\varepsilon} \, u^\varepsilon + (u^\varepsilon \cdot \mathfrak{grad}_\flat^\varepsilon) \, u^\varepsilon + \mathfrak{grad}_\flat^\varepsilon \, p^\varepsilon \; = \; \nu \; \varepsilon \; \Delta \, u^\varepsilon \, , \qquad \mathfrak{div}_\flat^\varepsilon \, u^\varepsilon = 0 \, , \\ u^\varepsilon(0, x, \theta) \; = \; u^\varepsilon_{(2,N)}(0, x, \theta) \, . \end{array} \right.$$

Theorem 5.1. *Fix any integer $N > d + 8$. There is $\varepsilon_N \in]0, 1]$ and $\nu_N > 0$ such that for all $\varepsilon \in]0, \varepsilon_N]$ and for all $\nu > \nu_N$ the Cauchy problem (5.4) has a unique solution $(u^\varepsilon, p^\varepsilon)$ defined on the strip $[0, T] \times \mathbb{R}^d \times \mathbb{T}$. Moreover*

$$\big\{ u^\varepsilon - u^\varepsilon_{(2,N)} \big\}_\varepsilon \; = \; \bigcirc(\varepsilon^{\frac{N}{2} - d - 4}) \, .$$

<u>Proof</u> <u>of</u> <u>the</u> <u>Theorem</u> <u>5.1</u>. The system (5.4) amounts to the same thing as

$$(5.5) \qquad \left\{ \begin{array}{l} \partial_t \bar{u}^\varepsilon + (\bar{u}^\varepsilon \cdot \nabla) \bar{u}^\varepsilon + \operatorname{div} \langle u^{\varepsilon *} \otimes u^{\varepsilon *} \rangle + \nabla \bar{p}^\varepsilon = \nu \; \Delta_x \, \bar{u}^\varepsilon \, , \\ \partial_{0,\varepsilon} u^{\varepsilon *} + (\bar{u}^\varepsilon \cdot \mathfrak{grad}_\flat^\varepsilon) \, u^{\varepsilon *} + \varepsilon \; (u^{\varepsilon *} \cdot \nabla) \, \bar{u}^\varepsilon \\ \qquad \quad + \; \big[(u^{\varepsilon *} \cdot \mathfrak{grad}_\flat^\varepsilon) \, u^{\varepsilon *} \big]^* + \mathfrak{grad}_\flat^\varepsilon \, p^{\varepsilon *} = \nu \; \varepsilon \; \Delta \, u^{\varepsilon *} \, , \\ \operatorname{div} \bar{u}^\varepsilon = \mathfrak{div}_\flat^\varepsilon \, u^{\varepsilon *} = 0 \, . \end{array} \right.$$

The equation (5.5) is also equivalent to solve the Cauchy problem

$$(5.6)\qquad \left\{ \begin{array}{l} P\,\partial_t \bar u^\varepsilon + P\ [(\bar u^\varepsilon\cdot\nabla)\bar u^\varepsilon] + P\ [\mathrm{div}\,\langle u^{\varepsilon *}\otimes u^{\varepsilon *}\rangle] = \nu\ \Delta_x\,\bar u^\varepsilon\,, \\ \mathfrak{P}^\varepsilon_\flat\,\partial_{0,\varepsilon} u^{\varepsilon *} + \mathfrak{P}^\varepsilon_\flat\ \big[(\bar u^\varepsilon\cdot \mathfrak{grad}^\varepsilon_\flat)\,u^{\varepsilon *}\big] + \varepsilon\ \mathfrak{P}^\varepsilon_\flat\ [(u^{\varepsilon *}\cdot\nabla)\,\bar u^\varepsilon] \\ \qquad\qquad + \mathfrak{P}^\varepsilon_\flat\ \big[(u^{\varepsilon *}\cdot \mathfrak{grad}^\varepsilon_\flat)\,u^{\varepsilon *}\big]^* = \nu\ \varepsilon\ \mathfrak{P}^\varepsilon_\flat\ \Delta\,u^{\varepsilon *}\,, \end{array}\right.$$

associated with the compatible initial data

$$\bar u^\varepsilon(0,\cdot) = P\,\bar u^\varepsilon_\flat(0,\cdot)\,,\qquad u^{\varepsilon *}(0,\cdot) = \mathfrak{P}^\varepsilon_\flat\,u^{\varepsilon *}_\flat(0,\cdot)\,.$$

• **Blow up.** Introduce the new unknown

$$\begin{aligned} d^\varepsilon\ &=\ {}^t(\bar d^\varepsilon, d^{\varepsilon *})\ =\ {}^t(\,P\,\bar d^\varepsilon\,,\,\mathfrak{P}^\varepsilon_\flat\,d^{\varepsilon *}\,) \\ &:=\ \varepsilon^{-\iota}\ \big(\varepsilon^{-\frac{1}{l}}\ (\bar u^\varepsilon - \bar u^\varepsilon_\flat)\,,\,(u^{\varepsilon *} - u^{\varepsilon *}_\flat)\big)\,,\qquad \flat = (2,N)\,. \end{aligned}$$

This transformation agrees with (3.5). The weight $\varepsilon^{-\frac{1}{l}}$ in front of $(\bar u^\varepsilon - \bar u^\varepsilon_\flat)$ induces a shift on the indice l. Functions $\bar U_l$ and U^*_{l-1} play now the same part related to the amplifications. To write the equation on d^ε in an abbreviated form, we need notations. Quasilinear terms

$$\begin{aligned} &\mathcal{L}^\varepsilon_{11}\,\bar d\ :=\ P\,\big[\,(\bar u^\varepsilon_\flat\cdot\nabla)\bar d\,\big]\,, \\ &\mathcal{L}^\varepsilon_{12}\,d^*\ :=\ P\,\big[\,\mathrm{div}\,\langle\varepsilon^{-\frac12}\ u^{\varepsilon *}_\flat\otimes d^*\ +\ d^*\otimes\varepsilon^{-\frac12}\ u^{\varepsilon *}_\flat\rangle\,\big]\,, \\ &\mathcal{L}^\varepsilon_{21}\,\bar d\ :=\ \varepsilon^{\frac12}\ \mathfrak{P}^\varepsilon_\flat\ \big[\,(u^{\varepsilon *}_\flat\cdot\nabla)\bar d\,\big]\,, \\ &\mathcal{L}^\varepsilon_{22}\,d^*\ :=\ \mathfrak{P}^\varepsilon_\flat\ \big[\,(\bar u^\varepsilon_\flat\cdot\nabla)d^*\,\big]\ +\ \varepsilon^{-1}\ \mathfrak{P}^\varepsilon_\flat\ \big[\,(u^{\varepsilon *}_\flat\cdot\mathfrak{grad}^\varepsilon_\flat)\,d^*\,\big]^* \\ &\qquad\qquad +\varepsilon^{-1}\ \mathfrak{P}^\varepsilon_\flat\ \big[\,(\partial_t\varphi^\varepsilon_\flat + \bar u^\varepsilon_\flat\cdot\nabla\varphi^\varepsilon_\flat)\ \partial_\theta d^*\,\big]\,. \end{aligned}$$

Semilinear terms

$$\begin{aligned} &A^\varepsilon_{11}\,\bar d\ :=\ P\,\big[\,(\bar d\cdot\nabla)\bar u^\varepsilon_\flat\,\big]\,, \\ &A^\varepsilon_{21}\,\bar d\ :=\ \mathfrak{P}^\varepsilon_\flat\,\big[\,(\bar d\cdot\mathfrak{grad}^\varepsilon_\flat)\,(\varepsilon^{-\frac12}\ u^{\varepsilon *}_\flat)\,\big]\,, \\ &A^\varepsilon_{22}\,d^*\ :=\ \mathfrak{P}^\varepsilon_\flat\ \big[\,(d^*\cdot\nabla)\,\bar u^\varepsilon_\flat\,\big]\ +\ \varepsilon^{-1}\ \mathfrak{P}^\varepsilon_\flat\ \big[\,(d^*\cdot\mathfrak{grad}^\varepsilon_\flat)\,u^{\varepsilon *}_\flat\,\big]^*\,. \end{aligned}$$

Small quadratic terms

$$\begin{aligned} &Q^\varepsilon_1\ :=\ \varepsilon^{\frac32}\ P\,\big[\,\mathrm{div}\ (\bar d\otimes\bar d)\big]\ +\ \varepsilon^{\frac12}\ P\,\big[\,\mathrm{div}\ \langle d^*\otimes d^*\rangle\,\big]\,, \\ &Q^\varepsilon_2\ :=\ \varepsilon^{\frac12}\ \mathfrak{P}^\varepsilon_\flat\ \big[\,(\bar d\cdot\mathfrak{grad}^\varepsilon_\flat)\,d^*\,\big]\ +\ \varepsilon^{\frac32}\ \mathfrak{P}^\varepsilon_\flat\ \big[\,(d^*\cdot\nabla)\,\bar d\,\big] \\ &\qquad\quad +\mathfrak{P}^\varepsilon_\flat\ \big[\,(d^*\cdot\mathfrak{grad}^\varepsilon_\flat)\,d^*\,\big]^*\,. \end{aligned}$$

And error terms

$$er^\varepsilon_1\ :=\ \varepsilon^{-\iota-\frac32}\ P\ \bar f^\varepsilon_\flat\,,\qquad er^\varepsilon_2\ :=\ \varepsilon^{-\iota-1}\ \mathfrak{P}^\varepsilon_\flat\ f^{\varepsilon *}_\flat\,.$$

With these conventions, the expression d^ε is subjected to

$$(5.7)\qquad \left\{ \begin{array}{l} P\,\partial_t\bar d^\varepsilon + \mathcal{L}^\varepsilon_{11}\,\bar d^\varepsilon + \mathcal{L}^\varepsilon_{12}\,d^{\varepsilon *} + A^\varepsilon_{11}\,\bar d^\varepsilon \\ \qquad\qquad +\,\varepsilon^{\iota-1}\ Q^\varepsilon_1 + er^\varepsilon_1\ =\ \nu\ P\ \Delta_x\bar d^\varepsilon\,, \\ \mathfrak{P}^\varepsilon_\flat\,\partial_t d^{\varepsilon *} + \mathcal{L}^\varepsilon_{21}\,\bar d^\varepsilon + \mathcal{L}^\varepsilon_{22}\,d^{\varepsilon *} + A^\varepsilon_{21}\,\bar d^\varepsilon + A^\varepsilon_{22}\,d^{\varepsilon *} \\ \qquad\qquad +\,\varepsilon^{\iota-1}\ Q^\varepsilon_2 + er^\varepsilon_2\ =\ \nu\ \mathfrak{P}^\varepsilon_\flat\ \Delta\,d^{\varepsilon *}\,. \end{array}\right.$$

Energy estimates are obtained at the level of (5.7). Below, we just sketch the related arguments which are classical.

• **L^2- estimates for the linear problem.** The linearized equations of Euler equations along the approximate solution $u^\varepsilon_\flat$ are obtained by removing Q^ε_1 and Q^ε_2 from (5.7). It yields a system which, at first sight, involves coefficients which are singular in ε. In fact, this is not the case. Let us explain why.

This is clear for $\mathcal{L}^\varepsilon_{11}$, $\mathcal{L}^\varepsilon_{21}$ and A^ε_{11}.

Since $u^{\varepsilon *}_\flat = \bigcirc(\varepsilon^{\frac{1}{l}})$, this is also true for $\mathcal{L}^\varepsilon_{12}$ and A^ε_{21}.

The contributions which in $\mathcal{L}^\varepsilon_{22}$ have ε^{-1} in factor give no trouble since

$$\partial_t \varphi^\varepsilon_\flat + \bar{u}^\varepsilon_\flat \cdot \nabla \varphi^\varepsilon_\flat = \bigcirc(\varepsilon^{\frac{N}{2}}) = \bigcirc(\varepsilon^{d+4}) \,, \qquad u^{\varepsilon *}_\flat \cdot \nabla \varphi^\varepsilon_\flat = v^{\varepsilon *}_\flat = \bigcirc(\varepsilon^{1+\frac{1}{l}}) \,.$$

Now, look at A^ε_{22}. Recall that $d^{\varepsilon *} = \mathfrak{P}^\varepsilon_\flat \, d^{\varepsilon *}$ which means that

$$\varepsilon^{-1} \, d^{\varepsilon *} \cdot \nabla \varphi^\varepsilon_\flat \; = \; - \operatorname{div} d^{\varepsilon *} \,.$$

Therefore

$$\varepsilon^{-1} \, \mathfrak{P}^\varepsilon_\flat \, \big[\, (d^{\varepsilon *} \cdot \mathfrak{grad}^\varepsilon_\flat) \, u^{\varepsilon *}_\flat \, \big]^* \; = \; T^\varepsilon(t,x,\nabla) \, d^{\varepsilon *} \,,$$

where T^ε is some differential operator of order 1 with bounded coefficients.

Observe that these manipulations and the blow up procedure induce a *loss* of hyperbolicity. When $\nu = 0$, this is the source of hidden instabilities. When $\nu \geq \nu_N > 0$ with ν_N large enough, this can be compensated by the viscosity. This is the key to L^2- estimates.

• **The non linear problem and higher order estimates.** Let σ be the smaller integer such that $\sigma \geq \frac{d+3}{2}$. If the life span T_ε of the exact solution u^ε is finite, we must have

$$\lim_{t \longrightarrow T_\varepsilon} \; \| \, u^\varepsilon(t,\cdot) \, \|_{H^\sigma} = \, +\infty \,.$$

Thus, the Theorem 5.1 is a consequence of the following majoration

$$\sup \, \big\{ \, \| \, u^\varepsilon(t,\cdot) \, \|_{H^\sigma} \, ; \; t \in [0, \min{(T_\varepsilon, T)}] \, \big\} \; \leq \; C \; < \; \infty \,.$$

Consider the set

$$\mathcal{Z}_\varepsilon \; := \; \big\{ \, \mathfrak{d}_{0,\varepsilon} \, , \, \cdots \, , \, \mathfrak{d}_{d,\varepsilon} \, , \, \partial_\theta \, \big\} \,.$$

Extract the operators

$$\mathcal{Z}^k_\varepsilon \; := \; \mathcal{Z}_1 \circ \, \cdots \, \circ \mathcal{Z}_k \,, \qquad \mathcal{Z}_j \in \mathcal{Z}_\varepsilon \,, \qquad k \leq \sigma \,.$$

It suffices to show that

$$\max_{0\leq k\leq\sigma} \quad \sup \, \big\{ \, \| \, \varepsilon^{-k} \ \mathcal{Z}_\varepsilon^k \ u^\varepsilon(t,\cdot) \, \|_{L^2} \, ; \ t\in[0,\min(T_\varepsilon,T)] \, \big\} \, \leq \, C \, < \, \infty \, .$$

Pick some $\mathcal{Z}_\varepsilon^k$ with $k\leq\sigma$. Apply $\mathcal{Z}_\varepsilon^k$ on the left of (5.7). Use the point i) of Lemma 4.2 to pass through $\mathfrak{P}_\flat^\varepsilon$. Then, observe that the commutator of two vector fields in $\mathcal{Z}^\varepsilon$ is a linear combination of elements of $\mathcal{Z}^\varepsilon$ with coefficients in C^∞. Thus, we get an equation on $\mathcal{Z}_\varepsilon^k \, d^{\varepsilon *}$.

The linear part is managed as in the preceding paragraph. Take $\iota=1$. The contributions due to Q_1^ε and Q_2^ε are controled by way of the a priori estimate and the viscosity. The condition on N is to make sure that

$$\tfrac{N}{2}-\iota-\tfrac{3}{2}-\sigma \, \geq \, 0 \, .$$

Thereby, the contributions brought by the error terms er_1^ε and er_2^ε remain bounded in the procedure.

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